

# USAMTS 3/2/35

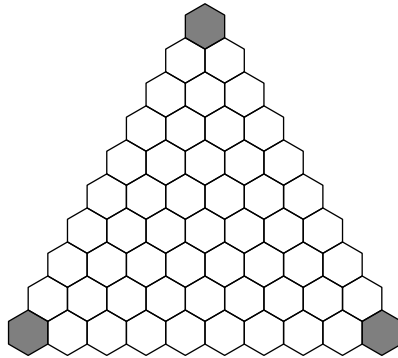
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TWITCH SOLVES ISL

Episode 136

## Problem

We say that three numbers are balanced if either all three numbers are the same, or they are all different. Consider the following hexagonal grid of side length 10:



Each hexagon is filled with the number 1, 2, or 3, so that in every row except the last, given any cell, it is balanced with the two entries in the cells below it. Prove that when the grid is filled completely, the three numbers in the three shaded hexagons are balanced.

## Video

<https://youtu.be/eFTzga0UNXw>

## External Link

<https://aops.com/community/p29309321>

## Solution

We need the following key observation:

**Lemma** (Known by people who play too much SET). Three numbers  $x, y, z \in \{1, 2, 3\}$  are balanced if and only if

$$x + y + z \equiv 0 \pmod{3}.$$

Work modulo 3 and let the bottom row be

$$a_0, \dots, a_9$$

in order. Then the second-to-bottom row (in order) is

$$-(a_0 + a_1), -(a_1 + a_2), \dots, -(a_8 + a_9).$$

and the third-to-bottom row is

$$a_0 + 2a_1 + a_2, a_1 + 2a_2 + a_3, \dots, a_7 + 2a_8 + a_9.$$

and so on; the pattern is given by binomial coefficients. One can then check the entry in the top row

$$-\sum_{k=0}^9 \binom{9}{k} a_k \equiv -(a_0 + a_9) \pmod{3}$$

since every binomial coefficient  $\binom{9}{k}$  is divisible by 3 for  $0 < k < 9$ , which implies the result.