HMNT 2023 Gen10

Evan Chen

Twitch Solves ISL

Episode 136

Problem

Let $ABCD$ be a convex trapezoid such that $\angle ABC = \angle BCD = 90^\circ$, $AB = 3$, $BC = 6$, and $CD = 12$. Among all points X inside the trapezoid satisfying $\angle XBC = \angle XDA$, compute the minimum possible value of CX.

Video

https://youtu.be/xR7rzNMpQlw

External Link

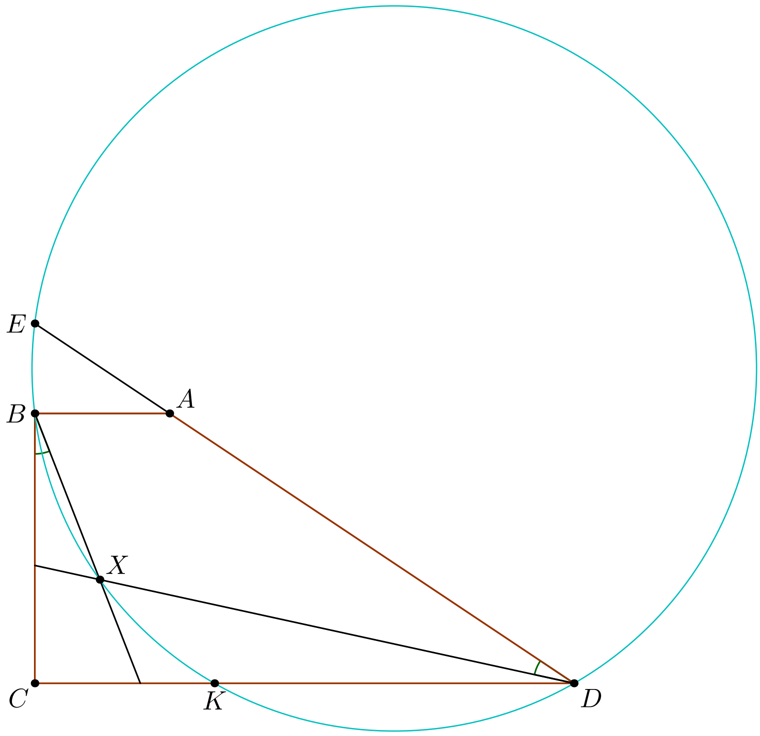
https://www.hmmt.org/www/archive/271

Solution

Let $E = \overline{BC} \cap \overline{AD}$.

Claim. The point X moves along the circumcircle of $\triangle EBD$.

Proof. Because $\angle XBE = 180^\circ - \angle CBX = 180^\circ - \angle ADX = 180^\circ - \angle EDX$. $\square$



It remains to do the calculation, which is surprisingly quick. Work in a coordinate system with

$$\begin{aligned} C &= (0,0) \\ B &= (0,6) \\ D &= (12,0) \\ A &= (3,6) \\ E &= (0,8). \end{aligned}$$

By power of a point, the second intersection of (EBD) with CD has

$$CK = \frac{CB \cdot CE}{CD} = \frac{6 \cdot 8}{12} = 4.$$

In other words, $K = (4,0)$. So the circumcenter of $\triangle EBD$ is located at

$$O = (8,7).$$

The radius of the circle is then $\sqrt{65}$, while $CO = \sqrt{113}$. It follows the answer is $\sqrt{113} - \sqrt{65}$.