H3170234

Evan Chen

Twitch Solves ISL

Episode 136

Problem

Solve over $\mathbb{Z}$ the functional equation

$$f(2x + f(y)) + f(f(2x)) = y.$$

Video

https://youtu.be/nyKKvc87dg8

External Link

https://aops.com/community/p28861167

Solution

The answers are

$$f(x) = \begin{cases} a - x & x \equiv 0 \pmod 2 \\ b - x & x \equiv 1 \pmod 2 \end{cases}$$

where a and b are either both even, or $a = b$. It can be checked that all of these work, so we prove they're the only solutions.

Let $P(x, y)$ be the given assertion.

- $P(0, 0) \implies f(f(0)) = 0$.
- $P(0, t) \implies f(f(t)) = t$.
- $P(1, f(z)) \implies f(z + 2) = z - 2$.

The last equation $f(z + 2) = z - 2$ implies f takes the above form for some a and b (in fact, $a = f(0)$ and $b = f(1) + 1$, but we don't need this particular expression). so we'd be done if we could show the parity condition. If a is odd, then plug in $x = 0$ to deduce $a = b$; if b is odd, plug in $x = 1$ to deduce $b = a$. This finishes the problem.