Twitch 133.6

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Twitch Solves ISL

Episode 133

Problem

Let ABC be a triangle and let Z be the A-Dumpty point. Let D, E, F be the reflections of Z over BC, AC, AB, respectively. Let the nine-point circles of ABC and DEF intersect at X, Y. Show that A, Z, X, Y are cyclic.

Video

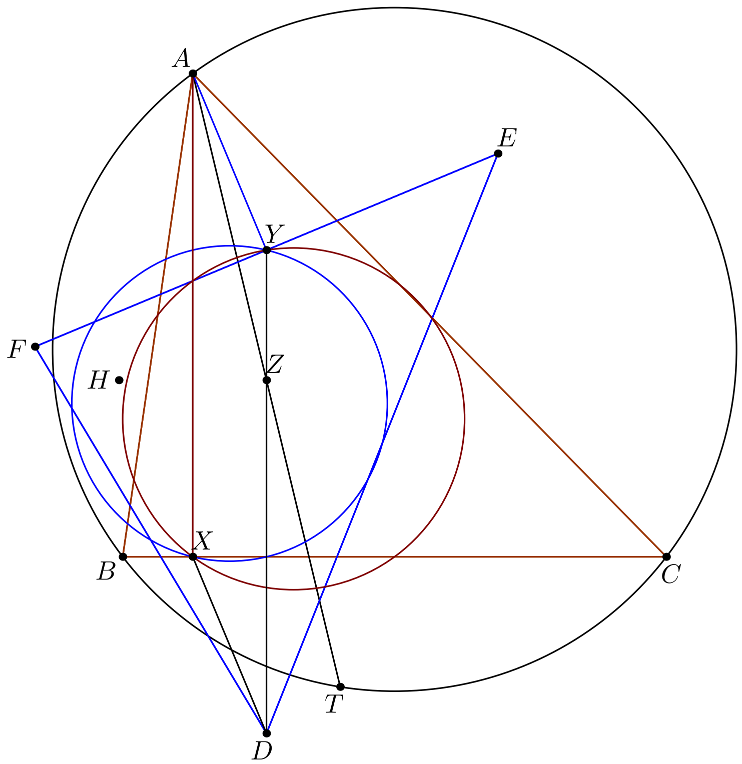
https://youtu.be/xitwCD-EOu0

External Link

https://aops.com/community/p28667211

Solution

We will use complex numbers with ABC the unit circle.



Let H be the reflection of Z across the A-altitude. We compute a bunch of complex numbers:

$$\begin{aligned}
0 &= (t-b)(a-c)+(t-c)(a-b) = (2a-b-c)t-(b(a-c)+c(a-b)) \\
\implies t &= \frac{b(a-c)+c(a-b)}{2a-b-c}. \\
\implies z &= \frac{a(2a-b-c)+b(a-c)+c(a-b)}{2(2a-b-c)} = \frac{a^2-bc}{2a-b-c}. \\
\overline{z} &= \frac{bc-a^2}{a(2bc-a(b+c))} \\
1-b\overline{z} &= \frac{a(2bc-a(b+c))-b(bc-a^2)}{a(2bc-a(b+c))} = \frac{2abc-a^2c-b^2c}{a(2bc-a(b+c))} = \frac{-c(a-b)^2}{a(2bc-a(b+c))} \\
\frac{1}{a}-\overline{z} &= \frac{(2bc-a(b+c))-(bc-a^2)}{a(2bc-a(b+c))} = \frac{bc-ab-ac+a^2}{a(2bc-a(b+c))} = \frac{(a-b)(a-c)}{a(2bc-a(b+c))} \\
d &= b+c-bc\overline{z} \\
e &= c+a-ca\overline{z} = c+a-\frac{c(2bc-a^2)}{2bc-a(b+c)} = \frac{-ac^2+abc-a^2b}{2bc-a(b+c)} = \frac{a(bc-c^2-ab)}{2bc-a(b+c)} \\
f &= a+b-ab\overline{z} \\
h &= a-\frac{bc}{a}+bc\overline{z}
\end{aligned}$$

Now, the proof requires three main claims.

Claim. The point H coincides with the orthocenter of $\triangle DEF$. In particular, we can set X as the foot of the altitude from A, which will lie on both nine-point circles.

Proof. Note that

$$\frac{e-h}{d-f} = \frac{c - ca\overline{z} + \frac{bc}{a} - bc\overline{z}}{(c-a)(1-b\overline{z})} = \frac{c(a+b)\left(\frac{1}{a} - \overline{z}\right)}{(c-a)\left(1-b\overline{z}\right)} = \frac{a+b}{a-b}$$

which is the negative of its own conjugate. □

Claim. The midpoint Y of EF also lies on the nine-point circle of ABC.

Proof. Note that

$$e+f-(a+b+c) = a(1-b\overline{z}-c\overline{z}) = \frac{-c(a-b)^2 - c(bc-a^2)}{2bc - a(b+c)} = \frac{-bc\,(-2a+b+c)}{2bc-a(b+c)}$$

which is equal to its own conjugate. Therefore, the reflection of the orthocenter over Y lies on the unit circle. □

Claim. $AXDY$ is a parallelogram.

Proof. Where $Y = \frac{e+f}{2}$, note that

$$2d - (e+f) = b + c - 2a - (2bc - a(b+c)) \cdot \frac{bc - a^2}{a(2bc - a(b+c))} = a + b + c - \frac{bc}{a}$$

which is twice the displacement form $X = \frac{1}{2}(a+b+c-bc/a)$ to A. □

In particular, $AY = XD = XZ$, so that $AYZX$ is an isosceles trapezoid, and cyclic.