H3195318

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Twitch Solves ISL

Episode 133

Problem

Given two positive real numbers x, y such that $x + y < \frac{\pi}{2}$ and $\sin(2y) + \sin(2x + 2y) = 2\sin(2x)$, determine, with proof, the value of

$$\frac{\cos(x + y)\sin(y)}{\sin(x)}.$$

Video

https://youtu.be/5A5Yu65JV_U

External Link

https://aops.com/community/p29162889

Solution

First manipulate both sides of the given equation to get

$$\begin{aligned}
\sin(2x+2y)+\sin(2y) &= \sin\left((x+2y)+x\right)+\sin\left((x+2y)-x\right) \\
&= 2\sin(x+2y)\cos x \\
2\sin(2x) &= 4\sin x\cos x.
\end{aligned}$$

So the problem condition is equivalent to

$$\begin{aligned}
\sin(x+2y) &= 2\sin(x) \\
\sin\left((x+y)+y\right) &= 2\sin\left((x+y)-y\right) \\
\sin(x+y)\cos(y)+\cos(x+y)\sin(y) &= 2\sin(x+y)\cos(y)-2\cos(x+y)\sin(y) \\
3\cos(x+y)\sin(y) &= \sin(x+y)\cos(y).
\end{aligned}$$

On the other hand, we could also write

$$\sin(x+y)\cos(y)+\cos(x+y)\sin(y)=2\sin(x).$$

Put together the two most recent equations to eliminate the undesired $\sin(x+y)\cos(y)$ term gives

$$3\cos(x+y)\sin(y)=2\sin(x)-\cos(x+y)\sin(y) \implies \frac{\cos(x+y)\sin(y)}{\sin(x)}=\boxed{\frac{1}{2}}.$$