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Twitch Solves ISL

Episode 133

Problem

Let triangle $\triangle ABC$ have orthocenter H. Let $K = X(69)$ be its isotomic conjugate. Let U and V be the intersection of HA with the B-median and HK with the C-median respectively. Suppose that UV intersects AC at X, and that XK intersects BC at Y. Show that $\angle ABU = \angle CYX$.

Video

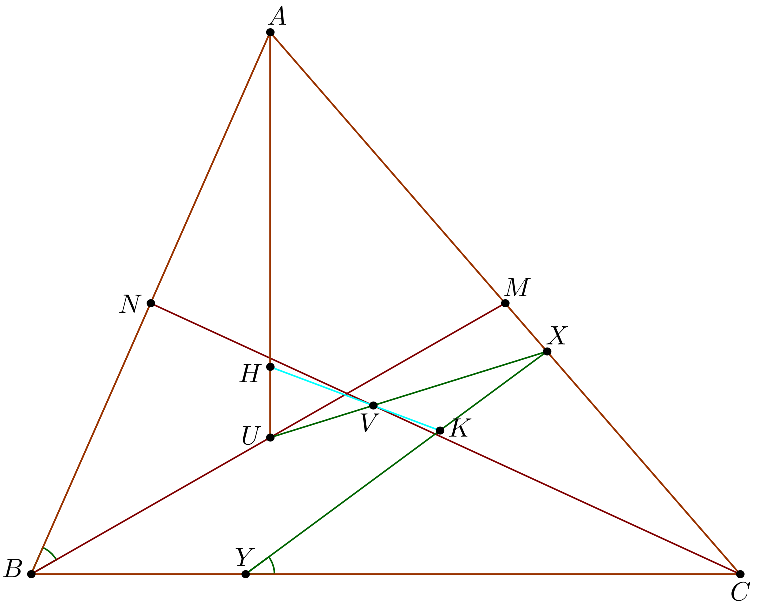
https://youtu.be/E3bzg5HKpCY

External Link

https://aops.com/community/p28831427

Solution

Let ∞ be the point at infinity along the B-symmedian. We will show that X, K, ∞ are collinear, solving the problem.



We use barycentric coordinates. The coordinates of all relevant points are:

$$
\begin{aligned}
H &= \left(\frac{1}{S_a} : \frac{1}{S_b} : \frac{1}{S_c}\right) \\
K &= (S_a : S_b : S_c) \\
U &= \left(\frac{1}{S_c} : \frac{1}{S_b} : \frac{1}{S_c}\right) = (S_b : S_c : S_b) \\
V &= (1 : 1 : v) \implies \left(\frac{S_b}{S_a} - \frac{S_a}{S_b}\right) v + \frac{S_c}{S_b} - \frac{S_b}{S_c} + \frac{S_a}{S_c} - \frac{S_c}{S_a} = 0
\end{aligned}
$$

$$
\implies (S_b^2 - S_a^2)v = \frac{S_a S_b (S_b - S_a)}{S_c} + S_c(S_b - S_a) \implies v = \frac{S_a S_b + S_c^2}{c^2 S_c}
$$

$$
\begin{aligned}
X &= (S_c - S_b : 0 : vS_c - S_b) = \left(b^2 - c^2 : 0 : \frac{S_a S_b - S_b c^2 + S_c^2}{c^2}\right) \\
&= \left(b^2 - c^2 : 0 : \frac{S_c^2 - S_b^2}{c^2}\right) \\
&= \left(c^2 : 0 : a^2\right) \\
\infty &= (a^2 : -(a^2 + c^2) : c^2).
\end{aligned}
$$

Now,

$$
\begin{aligned}
\det(\infty, X, K) = \det \begin{bmatrix} a^2 & -(a^2+c^2) & c^2 \\ c^2 & 0 & a^2 \\ S_a & S_b & S_c \end{bmatrix} &= \det \begin{bmatrix} a^2 & -(a^2+c^2) & c^2 \\ c^2 & 0 & a^2 \\ \frac{b^2+c^2}{2} & -\frac{b^2}{2} & \frac{a^2+b^2}{2} \end{bmatrix} \\
&= \det \begin{bmatrix} a^2 & -(a^2+c^2) & c^2 \\ c^2 & 0 & a^2 \\ \frac{b^2}{2} & -\frac{b^2}{2} & \frac{b^2}{2} \end{bmatrix} = 0.
\end{aligned}
$$