AMC 12A 2023/22

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Twitch Solves ISL

Episode 133

Problem

Let f be the unique function defined on the positive integers such that

$$\sum_{d|n} d \cdot f\left(\frac{n}{d}\right) = 1$$

for all positive integers n, where the sum is taken over all positive divisors of n. What is $f(2023)$?

Video

https://youtu.be/aqhUZkRuI4k

External Link

https://aops.com/community/p29157255

Solution

In the language of Dirichlet convolution, f is a function such that

$$\mathrm{id} * f = \mathbf{1}.$$

hence f is a multiplicative function.

It's easy to compute $f(1) = 1$ and $f(p^k) = 1 - p$ for primes p and $k \geq 1$. Since $2023 = 7 \cdot 17^2$,

$$f(2023) = f(7)f(289) = (-6)(-16) = \boxed{96}.$$