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TWITCH SOLVES ISL

Episode 132

## Problem

Let  $ABCD$  be a convex cyclic quadrilateral with diagonals intersecting at  $K$ . Let  $I_1, I_2, I_3, I_4$  denote the incenters of triangles  $ABK, BCK, CDK, DAK$ , and let  $M_1, M_2, M_3, M_4$  as the arc midpoints of the arcs  $AB, BC, CD, DA$  not containing any vertices of the quadrilateral in their interiors. Then prove that  $M_i I_i$  all concur at one point.

## Video

<https://youtu.be/luK5T-k08Ic>

## External Link

<https://aops.com/community/p2014625>

## Solution

In fact, letting  $O$  denote the center of the cyclic quadrilateral, we show the four lines concur on line  $OK$ . For this, it suffices to compute the coordinates of  $\overline{OK} \cap \overline{M_1I_1}$  and appeal to a suitable symmetry.

Let's set up a complex number system with  $A = a^2$ ,  $B = b^2$ ,  $C = c^2$ ,  $D = d^2$ , such that

$$\begin{aligned} K &= \frac{a^2c^2(b^2 + d^2) - b^2d^2(a^2 + c^2)}{a^2c^2 - b^2d^2} \\ \overline{K} &= \frac{a^2 + c^2 - (d^2 + b^2)}{a^2c^2 - b^2d^2} \\ M_1 &= -ab \\ M_3 &= +cd \\ M_2 &= -bc \\ M_4 &= -da. \end{aligned}$$

Then we can compute  $I_1$  as

$$\begin{aligned} I_1 &= \overline{AM_2} \cap \overline{BM_4} = \frac{a^2(-bc)(b^2 - ad) - b^2(-ad)(a^2 - bc)}{a^2(-bc) - b^2(-ad)} \\ &= \frac{bd(a^2 - bc) - ac(b^2 - ad)}{bd - ac}. \end{aligned}$$

Hence

$$\begin{aligned} M_1 - I_1 &= -ab - \frac{bd(a^2 - bc) - ac(b^2 - ad)}{bd - ac} \\ &= \frac{-ab(bd - ac) - bd(a^2 - bc) + ac(b^2 - ad)}{bd - ac} \\ &= \frac{(a + b)(abc - abd - acd + bcd)}{bd - ac} \end{aligned}$$

Now, note that

$$\begin{aligned} \overline{M_1I_1} \cap \overline{KO} &= \frac{(\overline{M_1I_1} - M_1\overline{I_1})(K - O) - (\overline{KO} - K\overline{O})(M_1 - I_1)}{(\overline{M_1} - \overline{I_1})(K - O) - (\overline{K} - \overline{O})(M_1 - I_1)} \\ &= K \cdot \frac{\overline{M_1I_1} - M_1\overline{I_1}}{(\overline{M_1} - \overline{I_1})(K) - (\overline{K})(M_1 - I_1)} \\ &= K \cdot \frac{-\frac{1}{ab} \cdot \frac{bd(a^2 - bc) - ac(b^2 - ad)}{bd - ac} + ab \cdot \frac{ac \cdot \left(\frac{1}{a^2} - \frac{1}{bc}\right) - bd \cdot \left(\frac{1}{b^2} - \frac{1}{ad}\right)}{ac - bd}}{\frac{(1/a + 1/b) \cdot (a + d - b - c)}{ac - bd} \cdot K - \frac{(a + b)(abc - abd - acd + bcd)}{bd - ac} \cdot \overline{K}} \\ &= \frac{K}{a + b} \cdot \frac{\frac{bd(a^2 - bc) - ac(b^2 - ad)}{ab} + ab \cdot \frac{ac \cdot \left(\frac{1}{a^2} - \frac{1}{bc}\right) - bd \cdot \left(\frac{1}{b^2} - \frac{1}{ad}\right)}{1}}{\frac{(a + d - b - c)}{ab} \cdot K + \frac{abc - abd - acd + bcd}{1} \cdot \overline{K}}. \end{aligned}$$

We focus on the latter fraction. The numerator is short enough to be computed first:

$$\frac{bd(a^2 - bc) - ac(b^2 - ad)}{ab} + ab \cdot \frac{ac \cdot \left(\frac{1}{a^2} - \frac{1}{bc}\right) - bd \cdot \left(\frac{1}{b^2} - \frac{1}{ad}\right)}{1} = -\frac{(a - b)(a + b)(ab - cd)}{ab}.$$

The denominator is more painful, but because we know the numerator we know some of the factors that must appear, and divide those out:

$$\begin{aligned} & \frac{(a+d-b-c)}{ab} \cdot K + \frac{abc - abd - acd + bcd}{1} \cdot \overline{K} \\ &= \frac{(a-b)(ab-cd) [a^2bc - a^2bd - a^2cd + ab^2c - ab^2d + abc^2 - abd^2 + ac^2d - acd^2 + b^2cd + bc^2d - bcd^2]}{ab(a^2c^2 - b^2d^2)} \end{aligned}$$

Thus, simplifying gives

$$\begin{aligned} & \overline{M_1I_1} \cap \overline{KO} \\ &= K \cdot \frac{a^2c^2 - b^2d^2}{a^2bc - a^2bd - a^2cd + ab^2c - ab^2d + abc^2 - abd^2 + ac^2d - acd^2 + b^2cd + bc^2d - bcd^2}. \end{aligned}$$

The symmetry of this expression is enough to conclude.