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Twitch Solves ISL

Episode 132

Problem

Let n be a positive integer. Let S be a multiset of $2n+1$ numbers less or equal than 2^n with the following property: the product of any n numbers of S divides the product of the $n+1$ remaining numbers of S. Prove that S has at least 2 equal numbers.

Video

https://youtu.be/3X6YcQybomM

External Link

https://aops.com/community/p28630932

Solution

We start by analyzing a particular prime:

Claim. Let p be a fixed prime, and let

$$e_0 \le \dots \le e_{2n}$$

be the ν_p's of the elements of S. There exists an index k such that

$$e_{k+1} = e_{k+2} \dots = e_{k+2n-e_0}.$$

Proof. Write the above inequality as

$$\sum_{i=1}^{n} (e_n - e_i) + \sum_{i=n+1}^{2n} (e_i - e_n) \le e_0$$

where

$$\begin{aligned} 0 = e_n - e_n \le e_n - e_{n-1} \le e_n - e_{n-2} \le \dots \le e_n - e_1 \\ 0 \le e_{n+1} - e_n \le e_{n+2} - e_n \le \dots \le e_{2n} - e_n. \end{aligned}$$

If we set $A = \sum_{i=1}^{n} (e_n - e_i)$ and $B = \sum_{i=n+1}^{2n} (e_i - e_n)$, then in fact the smallest $n - A$ summands of A and the smallest $n - B$ summands of B must all be zero. These produce the $A + B \le 2n - e_0$ required indices. □

We now proceed in two cases, by induction on n. The base cases $n = 0, 1, 2$ are clear, so assume $n \ge 3$.

- Assume there is a prime p that doesn't divide every element of S (i.e. $e_0 = 0$ for some prime p in the claim's notation). Then by the above, in fact there is one element of S not divisible by p, say x, and all other elements of S have the same exponent of p. Pick any $y \in S$ with $y \ne x$ and apply induction hypothesis onto $\frac{1}{p} \cdot (S \setminus \{x, y\})$.
- Otherwise, assume we're in a situation where the primes $p_1, \dots, p_m$ appear with nonzero minimum multiplicities $u_1, \dots, u_m$, respectively. Obviously we may assume $\sum u_i \le n - 1$, otherwise even the GCD of all the elements is at least $2^{u_1 + \dots + u_m}$. In particular, $\max(u_i) \le n - 1$; also (as $u_i \ge 1$) we have $m \le n - 1$.

  Imagine the following algorithm. Write all elements of S on the blackboard. Then for $i = 1, \dots, m$, take the index promised by the claim for the prime p_i, and draw an X on all $u_i + 1$ elements not corresponding to those e_i's. If, after the algorithm ends, there are at least two numbers not X'ed out, we win.

  However, the total number of X's is $\sum(u_i + 1) \le (n - 1) + m = 2n - 2$.