IMO 1973/5

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Twitch Solves ISL

Episode 132

Problem

Let G be a group, each of whose elements is a nonconstant linear function from $\mathbb{R}$ to itself, whose operation is function composition. Suppose that all elements of G have a fixed point. Show that all elements of G share a fixed point.

Video

https://youtu.be/mqkpYly4eg8

External Link

https://aops.com/community/p357910

Solution

Note that the only element of G of the form $x \mapsto x + \lambda$ is the identity.

Now consider any two non-identity functions of G, say

$$f(x) = ax + b$$
$$g(x) = cx + d$$

so that $a \neq 1$ and $c \neq 1$. The (unique) fixed point of $ax + b$ is $\frac{b}{a-1}$ and the (unique) fixed point of $cx + d$ is $\frac{d}{c-1}$, so we just need to show

$$\frac{b}{a-1} = \frac{d}{c-1}.$$

Note the inverses $f^{-1}(x) = \frac{x-b}{a}$ and $g^{-1}(x) = \frac{x-d}{c}$ should also be in G. Then consider the composition

$$\begin{aligned} g(f(g^{-1}(f^{-1}(x)))) &= c \cdot \left[a \cdot \frac{\frac{x-b}{a} - d}{c} + b \right] + d \\ &= x - b - ad + bc + d. \end{aligned}$$

Since this is an element of G, as we commented before we must have

$$b - ad + bc + d = 0$$

which was what we wanted.