

ELMO SL 2023 G7

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TWITCH SOLVES ISL

Episode 132

Problem

Let \mathcal{E} be an ellipse with foci F_1 and F_2 , and let P be a point on \mathcal{E} . Suppose line PF_1 and PF_2 intersect \mathcal{E} again at distinct points A and B , and the tangents to \mathcal{E} at A and B intersect at point Q . Show that the midpoint of \overline{PQ} lies on the circumcircle of $\triangle PF_1F_2$.

Video

<https://youtu.be/S3z2LSt7zzY>

External Link

<https://aops.com/community/p28033735>

Solution

Let $\rho = PF_1 + PF_2 = AF_1 + AF_2 = BF_1 + BF_2$. We start with the following claim:

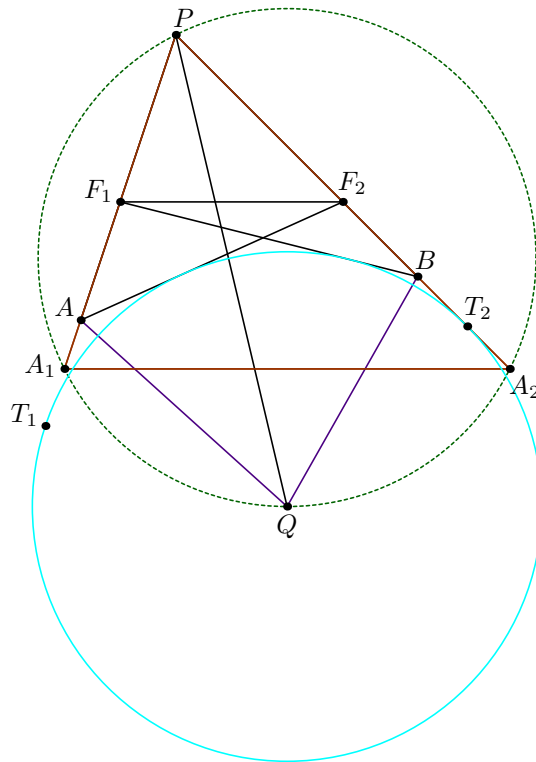
Claim. The P -excircles of triangle PAF_2 and PF_1B coincide.

Proof. This follows from the fact that $\triangle PAF_2$ and $\triangle PB_1$ have the same perimeter, namely 2ρ . \square

Claim. That common excircle has center Q .

Proof. The tangency of \overline{AQ} to the ellipse at the vertex of $\angle F_1AF_2$ implies that \overline{AQ} is the external bisector of $\angle F_1AF_2$ (by the famous “river problem”). Similarly, \overline{BQ} is the external bisector of $\angle F_2BF_1$. \square

Let A_1 and A_2 denote the reflections of P over F_1 and F_2 .



Claim. The points P, A_1, Q, A_2 are concyclic.

Proof. Let T_1 and T_2 denote the tangency points of (Q) to \overline{PA} and \overline{PB} . Note that

$$PT_1 + PT_2 = 2PT_1 = 2\rho = 2(PF_1 + PF_2) = PA_1 + PA_2 \implies T_1A_1 = T_2A_2$$

so it follows that $\triangle QT_1A_1 \cong \triangle QT_2A_2$ as right triangles, with the same orientation. In particular,

$$\angle PA_1Q = \angle T_1A_1Q = \angle T_2A_2Q = \angle T_1A_1Q = \angle P_2A_2Q. \quad \square$$

Finally, apply a homothety at P with ratio $\frac{1}{2}$ to deduce the problem statement.