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Twitch Solves ISL

Episode 131

Problem

Let ABC be a triangle with incircle ω. Let ω_1, ω_2, and ω_3 be three circles centered at A, B, and C respectively tangent to ω at points D, E, and F respectively. Show there exists a circle $\Gamma \neq \omega$ tangent to circles ω_1, ω_2, and ω_3 centered on the Euler line of $\triangle DEF$.

Video

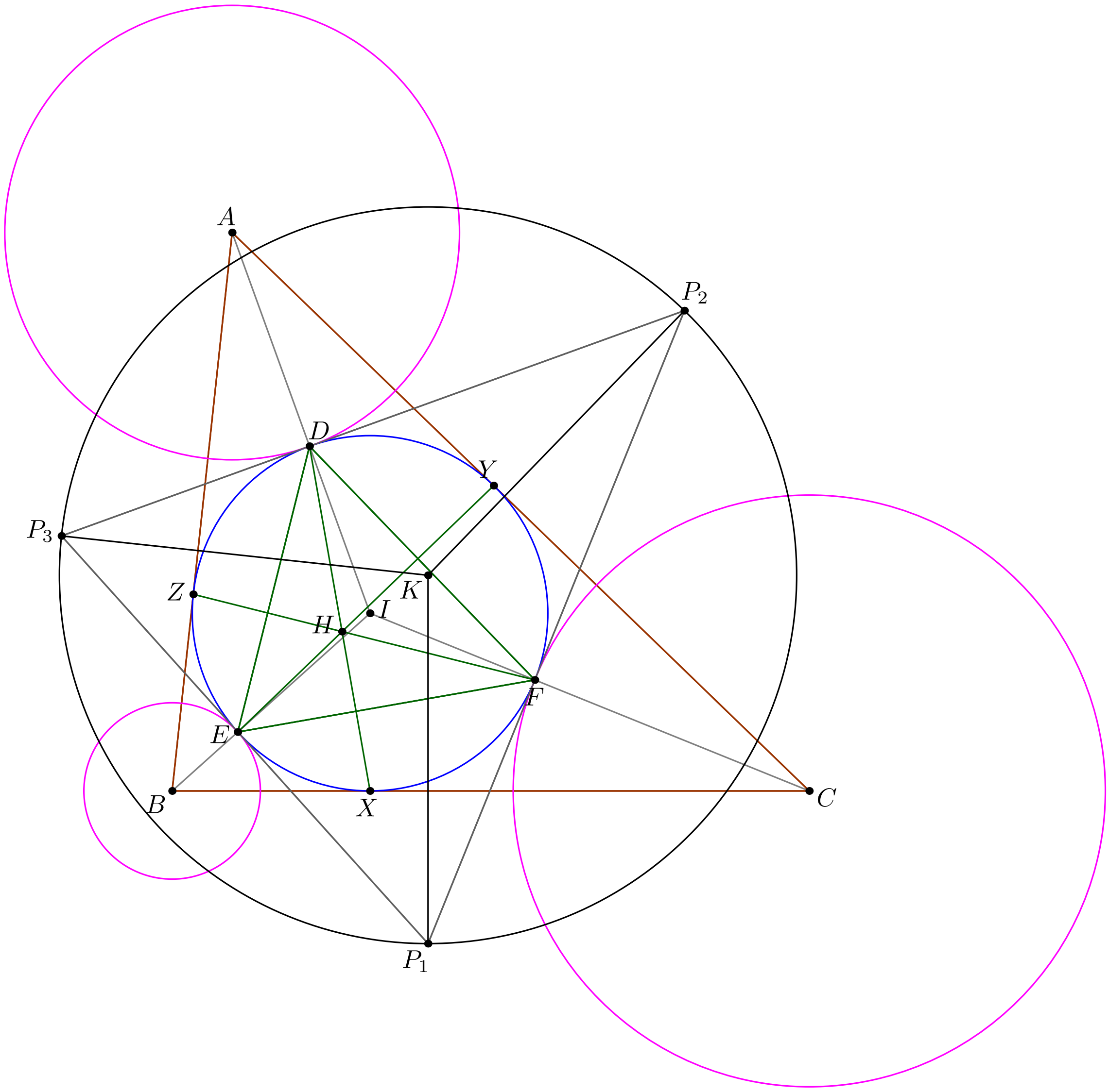
https://youtu.be/tGv-hnYaK0s

External Link

https://aops.com/community/p27839254

Solution

Let $\triangle XYZ$ be the intouch triangle of ABC. Let $\triangle P_1P_2P_3$ be so that $\triangle DEF$ is its intouch triangle. Let K be the radical center of ω_1, ω_2, ω_3.



Claim. $\overline{KP_1}$ is perpendicular to $\overline{BXC}$, etc.

Proof. P_1 lies on the radical axis of ω_2 and ω_3, so $\overline{P_1K}$ is their radical axis and perpendicular to the line through the centers B and C. □

Claim. K is actually the circumcenter of $\triangle P_1P_2P_3$.

Proof. Note that $\overline{BC}$ and $\overline{P_2P_3}$ are antiparallel with respect to $\angle P_1P_2P_3$, from $\overline{DX} \perp \overline{EF}$. □

Consequently, K lies on the Euler line of $\triangle DEF$, because the inverse of $(P_1P_2P_3)$ with respect to (DEF) is exactly the nine-point circle of $\triangle DEF$.

Finally, it's known that if $\Gamma \neq \omega$ is tangent to all three ω_i, then the line through the centers of Γ and ω passes through the radical center (in fact the radical center is either the insimilicenter or exsimilicenter). So that solves the problem.