Malaysia SST 2023/7

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Twitch Solves ISL

Episode 131

Problem

Find all polynomials with integer coefficients P such that for all positive integers n, the sequence

$$0, P(0), P(P(0)), P(P(P(0))), \ldots$$

is eventually constant modulo n.

Video

https://youtu.be/oNISgD_vjkM

External Link

https://aops.com/community/p28524876

Solution

The answer is:

- Any polynomial of the form $P(x) = Q(x)x(x-c) + c$, where $Q \in \mathbb{Z}[x]$ and $c \in \mathbb{Z}$.
- Any polynomial of the form $P(x) = Q(x)(x+c)(x-c) - c$, where $Q \in \mathbb{Z}[x]$ has $Q(0) = -\frac{2}{c}$ and $c \in \{\pm 1, \pm 2\}$.

The main claim of the problem is that the sequence can be described as follows.

Claim. The sequence is either

$$\begin{aligned} &0, c, \ c, \ c, \ \dots \\ &0, c, \ -c, \ -c, \ \dots \end{aligned}$$

for some integer c.

Proof. If $P(0) = 0$ we're done, so assume $c \coloneqq P(0) \neq 0$. Then every term must in the sequence is a multiple of c, by working modulo $|c|$.

On the other hand, I contend every term in the sequence is $\pm c$ now. Indeed, if some term M in the sequence is a different multiple of c, then take modulo $|M|$ to find the sequence is not eventually constant modulo M. (In the case $M = 0$, take modulo any number larger than c.) The claim then follows from this. $\square$

The answer then follows from the two cases of the claim. In the first case where the sequence is eventually constant, the only constraint is that we have $P(0) = P(c) = c$ for some c, which is equivalent to the bullet above. In the second case, we first read the constraint $P(c) = P(-c) = -c$ to get $P(x) = Q(x)(x+c)(x-c) - c$, and then plug in $x = 0$ to conclude.