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Twitch Solves ISL

Episode 131

Problem

Find all polynomials with integer coefficients ${\cal P}$ such that for all positive integers n, the sequence

 $0, P(0), P(P(0)), P(P(P(0))), \dots$

is eventually constant modulo n.

Video

https://youtu.be/oNISgD_vjkM

External Link

https://aops.com/community/p28524876

Solution

The answer is:

- Any polynomial of the form P(x) = Q(x)x(x-c) + c, where $Q \in \mathbb{Z}[x]$ and $c \in \mathbb{Z}$.
- Any polynomial of the form P(x) = Q(x)(x+c)(x-c) c, where $Q \in \mathbb{Z}[x]$ has $Q(0) = -\frac{2}{c}$ and $c \in \{\pm 1, \pm 2\}$.

The main claim of the problem is that the sequence can be described as follows.

Claim. The sequence is either

$$0, c, c, c, ...$$

 $0, c, -c, -c, ...$

for some integer c.

Proof. If P(0) = 0 we're done, so assume $c := P(0) \neq 0$. Then every term must in the sequence is a multiple of c, by working modulo |c|.

On the other hand, I contend every term in the sequence is $\pm c$ now. Indeed, if some term M in the sequence is a different multiple of c, then take modulo |M| to find the sequence is not eventually constant modulo M. (In the case M = 0, take modulo any number larger than c.) The claim then follows from this.

The answer then follows from the two cases of the claim. In the first case where the sequence is eventually constant, the only constraint is that we have P(0) = P(c) = c for some c, which is equivalent to the bullet above. In the second case, we first read the constraint P(c) = P(-c) = -c to get P(x) = Q(x)(x+c)(x-c) - c, and then plug in x = 0 to conclude.