Cross Ops

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Twitch Solves ISL

Episode 131

Problem

Given A, B, C, D, E on a line, define $f(P) = (A, B; C, P)$. Construct points X and Y with a straight edge such that $f(X) = f(D)f(E)$ and $f(Y) = f(D) + f(E)$.

Video

https://youtu.be/gjQyGxUu4YY

Solution

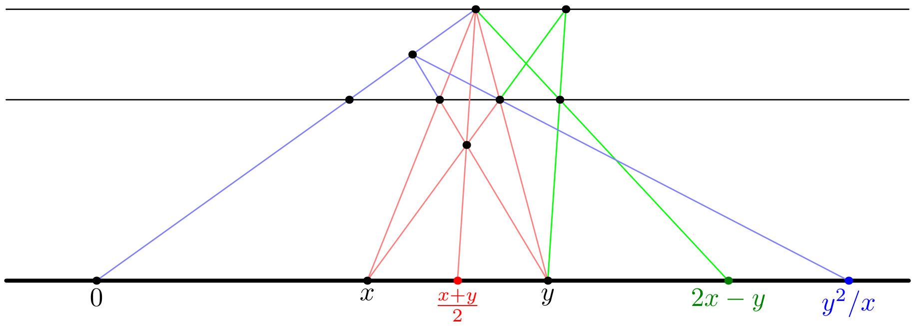
Impose a coordinate system where $A = \infty$, $B = 0$, $C = c$, $D = d$, $E = e$. In that case, if P has coordinate p, then

$$f(P) = \frac{CA}{CB} \div \frac{DA}{DB} = \frac{p}{c}.$$

We progressively give the following operations.

Claim. Given x and y, one can construct $\frac{x+y}{2}$, $2y - x$, and y^2/x.

Proof. Start by drawing any two other lines through $A = \infty$. See the following picture.



Follow red, green, blue in order. □

We are initially given points D and E such that $f(D) = d/c$ and $f(E) = e/c$. The point

$$X = 2\frac{d+e}{2} - 0 = d + e$$

thus has $f(X) = \frac{d+e}{c} = f(D) + f(E)$. Meanwhile, the point

$$Y = \frac{\frac{(d+e)^2}{c} - \frac{d^2}{c} - \frac{e^2}{c}}{2} = \frac{de}{c}$$

satisfies $f(Y) = \frac{de}{c^2} = f(D)f(E)$. We're done.