# IMO 2023/3

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Episode 130

## Problem

For each integer $k \geq 2$, determine all infinite sequences of positive integers $a_1$, $a_2$, ... for which there exists a polynomial $P$ of the form

$$P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0,$$

where $c_0$, $c_1$, ..., $c_{k-1}$ are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2} \cdots a_{n+k}$$

for every integer $n \geq 1$.

## Video

https://youtu.be/yGZUjUYTbzc

## External Link

https://aops.com/community/p28097600

## Solution

The answer is $a_n$ being an arithmetic progression. Indeed, if $a_n = d(n-1) + a_1$ for $d \geq 0$ and $n \geq 1$, then

$$a_{n+1}a_{n+2} \dots a_{n+k} = (a_n + d)(a_n + 2d) \dots (a_n + kd)$$

so we can just take $P(x) = (x+d)(x+2d) \dots (x+kd)$.

The converse direction takes a few parts.

**Claim.** Either $a_1 < a_2 < \cdots$ or the sequence is constant.

*Proof.* Note that

$$\begin{aligned} P(a_{n-1}) &= a_n a_{n+1} \cdots a_{n+k-1} \\ P(a_n) &= a_{n+1}a_{n+2} \cdots a_{n+k} \\ \implies a_{n+k} &= \frac{P(a_n)}{P(a_{n-1})} \cdot a_n. \end{aligned}$$

Now the polynomial $P$ is strictly increasing over $\mathbb{N}$.

So assume for contradiction there's an index $n$ such that $a_n < a_{n-1}$. Then in fact the above equation shows $a_{n+k} < a_n < a_{n-1}$. Then there's an index $\ell \in [n+1, n+k]$ such that $a_\ell < a_{\ell-1}$, and also $a_\ell < a_n$. Continuing in this way, we can an infinite descending subsequence of $(a_n)$, but that's impossible because we assumed integers.

Hence we have $a_1 \leq a_2 \leq \cdots$. Now similarly, if $a_n = a_{n-1}$ for any index $n$, then $a_{n+k} = a_n$, ergo $a_{n-1} = a_n = a_{n+1} = \cdots = a_{n+k}$. So the sequence is eventually constant, and then by downwards induction, it is fully constant. □

**Claim.** There exists a constant $C$ (depending only $P$, $k$) such that we have $a_{n+1} \leq a_n + C$.

*Proof.* Let $C$ be a constant such that $P(x) < x^k + Cx^{k-1}$ for all $x \in \mathbb{N}$ (for example $C = c_0 + c_1 + \cdots + c_{k-1} + 1$ works). We have

$$\begin{aligned} a_{n+k} &= \frac{P(a_n)}{a_{n+1}a_{n+2} \dots a_{n+k-1}} \\ &< \frac{P(a_n)}{(a_n+1)(a_n+2) \dots (a_n+k-1)} \\ &< \frac{a_n^k + C \cdot a_n^{k-1}}{(a_n+1)(a_n+2) \dots (a_n+k-1)} \\ &< a_n + C + 1. \end{aligned}$$

□

Assume henceforth $a_n$ is nonconstant, and hence unbounded. For each index $n$ and term $a_n$ in the sequence, consider the associated differences $d_1 = a_{n+1} - a_n$, $d_2 = a_{n+2} - a_{n+1}$, …, $d_k = a_{n+k} - a_{n+k-1}$, which we denote by

$$\Delta(n) \coloneqq (d_1, \dots, d_k).$$

This $\Delta$ can only take up to $C^k$ different values. So in particular, some tuple $(d_1, \dots, d_n)$ must appear infinitely often as $\Delta(n)$; for that tuple, we obtain

$$P(a_N) = (a_N + d_1)(a_N + d_1 + d_2) \dots (a_N + d_1 + \cdots + d_k)$$

for infinitely many $N$. But because of that, we actually must have

$$P(X) = (X + d_1)(X + d_1 + d_2) \dots (X + d_1 + \cdots + d_k).$$

However, this *also* means that *exactly* one output to $\Delta$ occurs infinitely often (because that output is determined by $P$). Consequently, it follows that $\Delta$ is eventually constant. For this to happen, $a_n$ must eventually coincide with an arithmetic progression of some common difference $d$, and $P(X) = (X+d)(X+2d)\dots(X+kd)$. Finally, this implies by downwards induction that $a_n$ is an arithmetic progression on all inputs.