IMC 2023/2

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Twitch Solves ISL

Episode 130

Problem

Let A, B and C be $n \times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \text{ and } B^3 = ABC + 2\,\mathrm{id}\,.$$

Prove that $A^6 = \mathrm{id}$.

Video

https://youtu.be/JKne3A1Pu8o

External Link

https://aops.com/community/p28295670

Solution

We start with the following claim:

Claim. We have $A = -C$ and $B^2 = -AC$.

Proof. We started with

$$\begin{aligned} B^5 &= ABCB^2 + 2B^2 = ABC^3 + 2C^2 = AB^3C + 2C^2 \\ &= A(ABC+2)C + 2C^2 = B^5 + 2AC + 2C^2 \\ \implies 0 &= AC + C^2. \end{aligned}$$

Following the analogous argument gives

$$A^2 = B^2 = C^2 = -AC.$$

Now we have

$$-BAC = -ACB = ABC + 2$$

Notice that this means that $-2 = (AB + BA)C$, so C is invertible. Combined with the claim that $AC + C^2 = 0$, this gives $A = -C$. □

Now squaring both sides of $B^3 = -ABA + 2$ gives

$$\begin{aligned} B^6 &= ABA^2BA - 4ABA + 4 = AB^4A + 4B^3 - 4 \\ &= A(A^2)(A^2)A + 4B^3 - 4 \\ &= A^6 + 4B^3 + 4 = B^6 + 4B^3 + 4 \end{aligned}$$

and hence $B^3 = -\operatorname{id}$, so $A^6 = B^6 = \operatorname{id}$.