ELMO SL 2023 G1

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Episode 130

Problem

Let $ABCDE$ be a regular pentagon. Let P be a variable point on the interior of segment AB such that $PA \neq PB$. The circumcircles of $\triangle PAE$ and $\triangle PBC$ meet again at Q. Let R be the circumcenter of $\triangle DPQ$. Show that as P varies, R lies on a fixed line.

Video

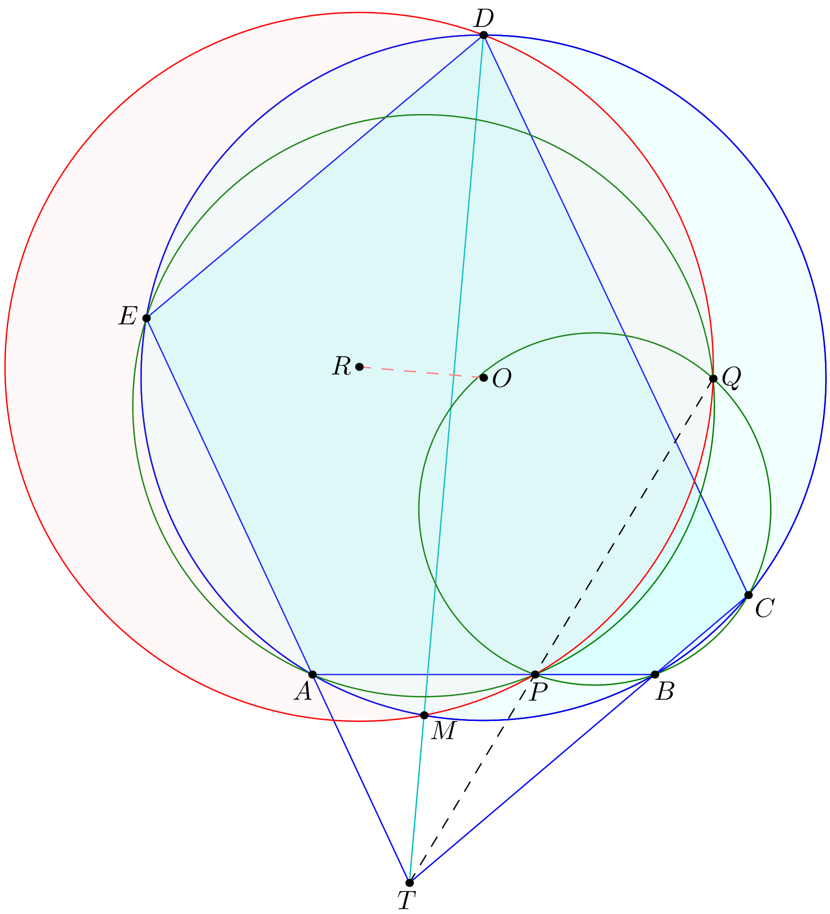
https://youtu.be/wMdc6hUhhaA

External Link

https://aops.com/community/p28033718

Solution

In fact, the condition that $ABCDE$ is regular is needlessly strong. The conclusion holds for any cyclic pentagon.



Claim. D, P, Q, M are cyclic.

Proof. Let $T := \overline{AE} \cap \overline{BC}$ and let $\overline{DT}$ meet the circumcircle of the pentagon again at M. Since $TA \cdot TE = TB \cdot TC$, it follows T lies on the radical axis $\overline{PQ}$. Then $TP \cdot TQ = TM \cdot TD$. □

Hence R lies on the perpendicular bisector of $\overline{DM}$, which does not depend on the choice of P and is thus the desired fixed line.

Remark. We also never used the fact that P lies on line AB.