IMO 2023/4

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Twitch Solves ISL

Episode 129

Problem

Let x_1, x_2, ..., x_{2023} be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geq 3034$.

Video

https://youtu.be/vDG6i7LmiFU

External Link

https://aops.com/community/p28104298

Solution

Note that for $a_{n+1} > \sqrt{\prod_1^n x_i \prod_1^n \frac{1}{x_i}} = a_n$ for all n. Now, the main claim is:

Claim. It's impossible to have $a_n = c$, $a_{n+1} = c+1$, $a_{n+2} = c+2$ for any c and n.

Proof. Let $p = x_{n+1}$ and $q = x_{n+2}$ for brevity. Let $s = \sum_1^n x_i$ and $t = \sum_1^n \frac{1}{x_n}$, so $c^2 = a_n^2 = st$.

From $a_n = c$ and $a_{n+1} = c$ we have

$$\begin{aligned}(c+1)^2 = a_{n+1}^2 &= (p+s)\left(\frac{1}{p}+t\right) \\ &= st + pt + \frac{1}{p}s + 1 = c^2 + pt + \frac{1}{p}s + 1 \\ &\overset{\text{AM-GM}}{\geq} c^2 + 2\sqrt{st} + 1 = c^2 + 2\sqrt{c^2} + 1 = (c+1)^2.\end{aligned}$$

Hence, equality must hold in the AM-GM we must have exactly

$$pt = \frac{1}{p}s = c.$$

If we repeat the argument again on $a_{n+1} = c+1$ and $a_{n+2} = c_{n+2}$, then

$$p\left(\frac{1}{q}+t\right) = \frac{1}{p}(q+s) = c+1.$$

However this forces $\frac{p}{q} = \frac{q}{p} = 1$ which is impossible. $\square$

To conclude, observe $a_1 = 1$ and $a_2 \geq 3$ (as $a_2 = 2$ would force $x_1 = x_2$). So now $a_{n+1} > a_n$ together with the claim are enough to show $a_{2m+1} \geq 3m$ for all m, as desired.