IMO 2023/4

Evan Chen

Twitch Solves ISL

Episode 129

Problem

Let x_1, x_2, $\ldots$, x_{2023} be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geq 3034$.

Video

https://youtu.be/vDG6i7LmiFU

External Link

https://aops.com/community/p28104298

Solution

Note that $a_{n+1} > \sqrt{\sum_1^n x_i \sum_1^n \frac{1}{x_i}} = a_n$ for all n, so that $a_{n+1} \geq a_n + 1$. Observe $a_1 = 1$. We are going to prove that

$$a_{2m+1} \geq 3m + 1 \qquad \text{for all } m \geq 0$$

by induction on m, with the base case being clear.

We now present two variations of the induction. The first shorter solution compares a_{n+2} directly to a_n, showing it increases by at least 3. Then we give a longer approach that compares a_{n+1} to a_n, and shows it cannot increase by 1 twice in a row.

Induct-by-two solution. Let $u = \sqrt{\frac{x_{n+1}}{x_{n+2}}} \neq 1$. Note that by using Cauchy-Schwarz with three terms:

$$\begin{aligned}
a_{n+2}^2 &= \left[(x_1 + \dots + x_n) + x_{n+1} + x_{n+2}\right]\left[\left(\frac{1}{x_1} + \dots + \frac{1}{x_n}\right) + \frac{1}{x_{n+2}} + \frac{1}{x_{n+1}}\right] \\
&\geq \left(\sqrt{(x_1 + \dots + x_n)\left(\frac{1}{x_1} + \dots + \frac{1}{x_n}\right)} + \sqrt{\frac{x_{n+1}}{x_{n+2}}} + \sqrt{\frac{x_{n+2}}{x_{n+1}}}\right)^2 \\
&= \left(a_n + u + \frac{1}{u}\right)^2. \\
\implies a_{n+2} &\geq a_n + u + \frac{1}{u} > a_n + 2
\end{aligned}$$

where the last equality $u + \frac{1}{u} > 2$ is by AM-GM, strict as $u \neq 1$. It follows that $a_{n+2} \geq a_n + 3$, completing the proof.

Induct-by-one solution. The main claim is:

Claim. It's impossible to have $a_n = c$, $a_{n+1} = c + 1$, $a_{n+2} = c + 2$ for any c and n.

Proof. Let $p = x_{n+1}$ and $q = x_{n+2}$ for brevity. Let $s = \sum_1^n x_i$ and $t = \sum_1^n \frac{1}{x_n}$, so $c^2 = a_n^2 = st$.

From $a_n = c$ and $a_{n+1} = c$ we have

$$\begin{aligned}
(c+1)^2 = a_{n+1}^2 &= (p + s)\left(\frac{1}{p} + t\right) \\
&= st + pt + \frac{1}{p}s + 1 = c^2 + pt + \frac{1}{p}s + 1 \\
&\overset{\text{AM-GM}}{\geq} c^2 + 2\sqrt{st} + 1 = c^2 + 2\sqrt{c^2} + 1 = (c+1)^2.
\end{aligned}$$

Hence, equality must hold in the AM-GM we must have exactly

$$pt = \frac{1}{p}s = c.$$

If we repeat the argument again on $a_{n+1} = c + 1$ and $a_{n+2} = c_{n+2}$, then

$$p\left(\frac{1}{q} + t\right) = \frac{1}{p}(q + s) = c + 1.$$

However this forces $\frac{p}{q} = \frac{q}{p} = 1$ which is impossible. $\square$