IMO 2023/2

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Twitch Solves ISL

Episode 129

Problem

Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

Video

https://youtu.be/Alz2OWHH7QY

External Link

https://aops.com/community/p28097552

Solution

Claim. We have LPS collinear.

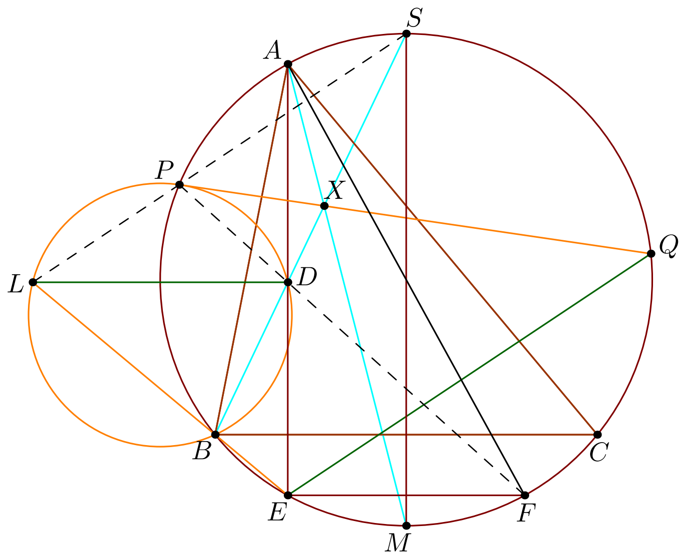
Proof. Because $\measuredangle LPB = \measuredangle LDB = \measuredangle CBD = \measuredangle CBS = \measuredangle SCB = \measuredangle SPB$. □

Let F be the antipode of A, so $AMFS$ is a rectangle.

Claim. We have PDF collinear. (This lets us erase L.)

Proof. Because $\measuredangle SPD = \measuredangle LPD = \measuredangle LBD = \measuredangle SBE = \measuredangle FCS = \measuredangle FPS$. □

Let us define $X = \overline{AM} \cap \overline{BS}$ and complete chord $\overline{PXQ}$. We aim to show that $\overline{PXQ}$ is tangent to $(PDLB)$.

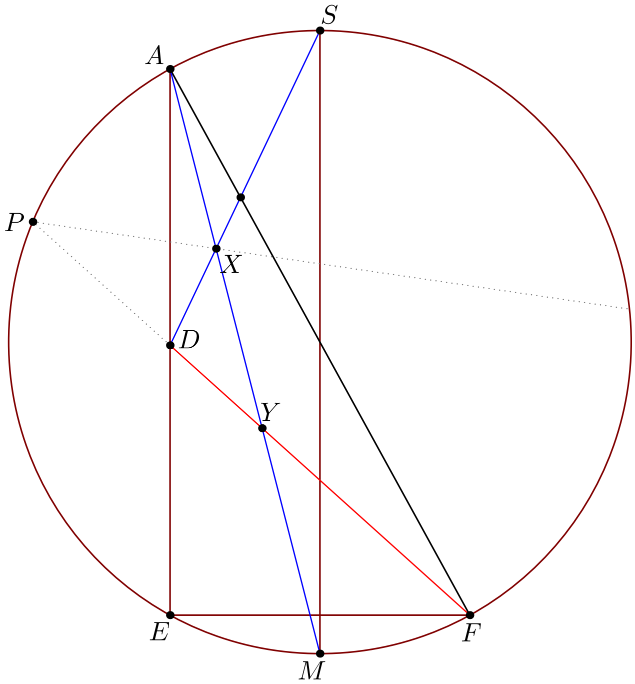


Claim (Main projective claim)**.** We have $XP = XA$.

Proof. Introduce $Y = \overline{PDF} \cap \overline{AM}$. Note that

$$-1 = (SM; EF) \overset{A}{=} (S, X; D, \overline{AF} \cap \overline{ES}) \overset{F}{=} (\infty X; YA)$$

where $\infty = \overline{AM} \cap \overline{SF}$ is at infinity (because $AMSF$ is a rectangle). Thus, $XY = XA$.



Since $\triangle APY$ is also right, we get $XP = XA$. $\square$

Alternative proof of claim without harmonic bundles, from Solution 9 of the marking scheme. With $Y = \overline{PDF} \cap \overline{AM}$ defined as before, note that $\overline{AE} \parallel \overline{SM}$ and $\overline{AM} \parallel \overline{SF}$ (as $AMFS$ is a rectangle) gives respectively the similar triangles

$$\triangle AXD \sim \triangle MXS, \qquad \triangle XDY \sim \triangle SDF.$$

From this we conclude

$$\frac{AX}{XD} = \frac{AX + XM}{XD + SX} = \frac{AM}{SD} = \frac{SF}{SD} = \frac{XY}{XD}.$$

So $AX = XY$ and as before we conclude $XP = XA$. $\square$

From $XP = XA$, we conclude that $\widehat{PM}$ and $\widehat{AQ}$ have the same measure. Since $\widehat{AS}$ and $\widehat{EM}$ have the same measure, it follows $\widehat{PE}$ and $\widehat{SQ}$ have the same measure. The desired tangency then follows from

$$\measuredangle QPL = \measuredangle QPS = \measuredangle PQE = \measuredangle PFE = \measuredangle PDL.$$

Remark (Logical ordering)**.** This solution is split into two phases: the "synthetic phase" where we do a bunch of angle chasing, and the "projective phase" where we use cross-ratios because I like projective. For logical readability (so we write in only one logical direction), the projective phase is squeezed in two halves of the synthetic phase, but during an actual solve it's expected to complete the whole synthetic phase first (i.e. to reduce the problem to show $XP = XA$).

Remark. There are quite a multitude of approaches for this problem; the marking scheme for this problem at the actual IMO had 13 different solutions.