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Twitch Solves ISL

Episode 129

Problem

Determine all composite integers $n > 1$ that satisfy the following property: if $d_1 < d_2 < \cdots < d_k$ are all the positive divisors of n with then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k-2$.

Video

https://youtu.be/Z7G7bSGfrqU

External Link

https://aops.com/community/p28097575

Solution

The answer is prime powers.

Verification that these work. When $n = p^e$, we get $d_i = p^{i-1}$. The i^{th} relationship reads

$$p^{i-1} \mid p^i + p^{i+1}$$

which is obviously true.

Proof that these are the only answers. Conversely, suppose n has at least two distinct prime divisors. Let $p < q$ denote the two smallest ones, and let p^e be the largest power of p which both divides n and is less than q, hence $e \geq 1$. Then the smallest factors of n are 1, p, $\dots$, p^e, q. So we are supposed to have

$$\frac{n}{q} \mid \frac{n}{p^e} + \frac{n}{p^{e-1}} = \frac{(p+1)n}{p^e}$$

which means that the ratio

$$\frac{q(p+1)}{p^e}$$

needs to be an integer, which is obviously not possible.