Twitch 125.4

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Twitch Solves ISL

Episode 125

Problem

In triangle ABC, let I be the incenter, and let D be the incircle touch point to BC. Consider the circle ω tangent to AI at I passing through D; it intersects the incircle again at a point X. Show that the other intersection of AX with the circumcircle also lies on ω.

Video

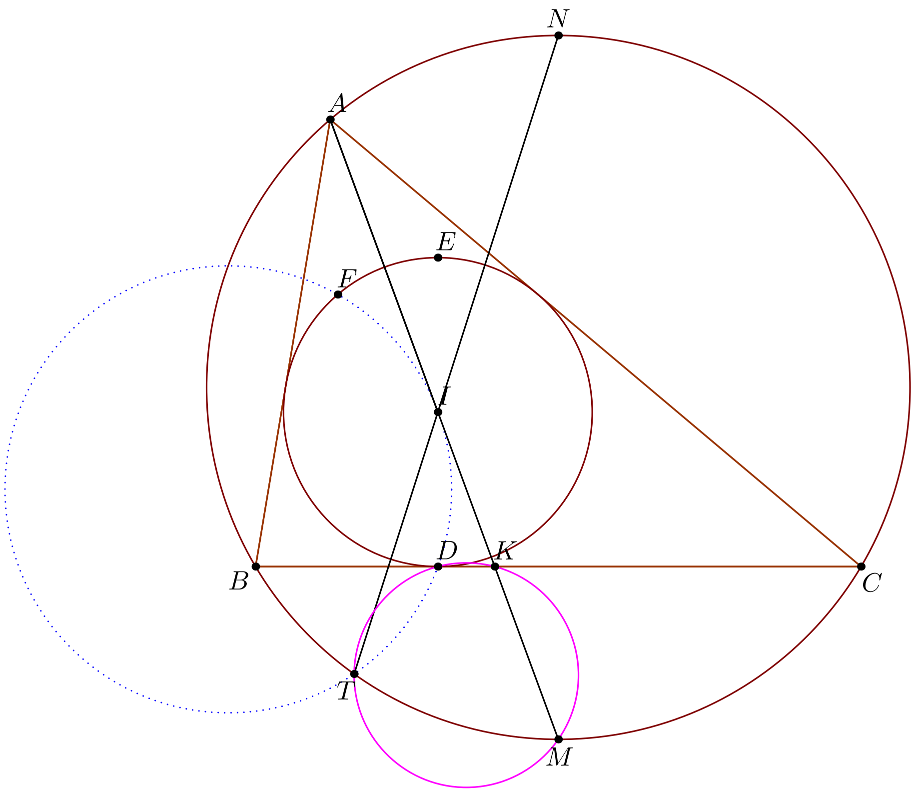
https://youtu.be/3Dd8R3u_8H6BM

Solution

We define many points:

- Let E be the antipode of D on the incircle.
- Let F be the reflection of E across $\overline{AI}$.
- Let M be the minor arc midpoint of $\overline{BC}$, and N the major arc midpoint.
- Let $K = \overline{AIM} \cap \overline{BC}$.
- Let T be the contact point of the A-mixtilinear incircle. We will use the following known properties of T:
  - $DKTM$ is cyclic,
  - T lies on line IN
  - $\overline{AT}$ is isogonal to the A-Nagel cevian.

See figure below.



There are two main claims in the problem.

Claim. (DIT) is tangent to $\overline{AIKM}$ at I.

Proof. By angle chasing, because

$$\measuredangle IDT = \measuredangle KDT + 90^\circ = \measuredangle KMT + 90^\circ = \frac{1}{2}\widehat{AT} + 90^\circ = \frac{1}{2}\widehat{AT} + \frac{1}{2}\widehat{MN} = \measuredangle AIT.$$

Here arcs are directed modulo 360°. □

Claim. (FID) is tangent to $\overline{AIKM}$ at I.

Proof. This is obvious from the fact that $IF = ID$ and $\overline{IF}$ and $\overline{ID}$ are reflections through line $\overline{AIKM}$. □

Claim. Points A, F, T are collinear.

Proof. $\overline{AF}$ is isogonal to $\overline{AE}$ because they're symmetric around line $\overline{AI}$. Meanwhile, $\overline{AE}$ is the A-Nagel cevian, which is isogonal to $\overline{AT}$. □

Hence, the point X in the problem statement is F, and the "other intersection" mentioned by the problem is T. We're done.