Twitch 125.2

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Twitch Solves ISL

Episode 125

Problem

Regular hexagon $ABCDEF$ is drawn, and a point P inside this hexagon satisfies $[PAB] = 6$, $[PCD] = 10$, and $[PEF] = 11$. Compute $[PBC][PDE][PFA]$.

Video

Solution

Let x denote the side length of the hexagon, so the heights from P to AB, CD, EF are $\frac{12}{x}$, $\frac{20}{x}$, $\frac{22}{x}$ respectively. Let h_1, h_2, h_3 be the heights from P to BC, DE, FA; then

$$\sqrt{3}x = \frac{12}{x} + h_1$$
$$= \frac{20}{x} + h_2$$
$$= \frac{22}{x} + h_3.$$

On the other hand, by considering the equilateral triangles formed by extending the sides of the hexagon (or quoting Vivani theorem),

$$h_1 + h_2 + h_3 = \frac{12}{x} + \frac{20}{x} + \frac{22}{x}.$$

Hence, it follows that

$$3 \cdot \sqrt{3}x = 2\left(\frac{12}{x} + \frac{20}{x} + \frac{22}{x}\right)$$

so $x^2 = 12\sqrt{3}$.

To finish up, compute

$$[PDE] = \frac{1}{2}x \cdot h_1 \qquad\qquad = \frac{\sqrt{3}}{2}x^2 - 6 = 12$$
$$[PFA] = \frac{1}{2}x \cdot h_2 \qquad\qquad = \frac{\sqrt{3}}{2}x^2 - 10 = 8$$
$$[PBC] = \frac{1}{2}x \cdot h_3 \qquad\qquad = \frac{\sqrt{3}}{2}x^2 - 11 = 7$$

giving answer $12 \cdot 8 \cdot 7 = 672$.