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Twitch Solves ISL

Episode 125

Problem

Find all functions $f \colon \mathbb{N} \to \mathbb{R}$ such that for all triples a, b, c of positive integers the following holds:

$$f(ac) + f(bc) - f(c)f(ab) \geq 1.$$

Video

https://youtu.be/3D5R2FHB0MVz0

External Link

https://aops.com/community/p22052363

Solution

The answer is $f \equiv 1$ only, which works. Conversely, let $P(a, b, c)$ be the statement.

- From $P(1, 1, 1)$ we get $0 \geq (f(1) - 1)^2 \implies \boxed{f(1) = 1}$
- From $P(1, 1, c)$ we get $f(c) \geq 1$.

Then, note that $P(x, x, x)$ gives the statement

$$\begin{aligned} f(x^2) + f(x^2) - f(x)f(x^2) &\geq 1 \\ \implies (2 - f(x))\, f(x^2) &\geq 1 \\ \implies 2 - f(x) &\geq \frac{1}{f(x^2)} \\ \implies f(x) &\leq 2 - \frac{1}{f(x^2)} \end{aligned}$$

since f only takes positive values.

In particular, this implies that

- we have $f(n) \leq 2$ for all $n \geq 1$;
- we have $f(n) \leq \frac{3}{2}$ for all $n \geq 1$ (by using the previous statement to get $f(n^2) \leq 2$);
- we have $f(n) \leq \frac{4}{3}$ for all $n \geq 1$ (again by previous);
- we have $f(n) \leq \frac{5}{4}$ for all $n \geq 1$ (again by previous);
- and so on.

It follows that $f(n) = 1$.

Remark. Note that the problem statement is only ever used when $a = b$.