

Twitch 124.3

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TWITCH SOLVES ISL

Episode 124

Problem

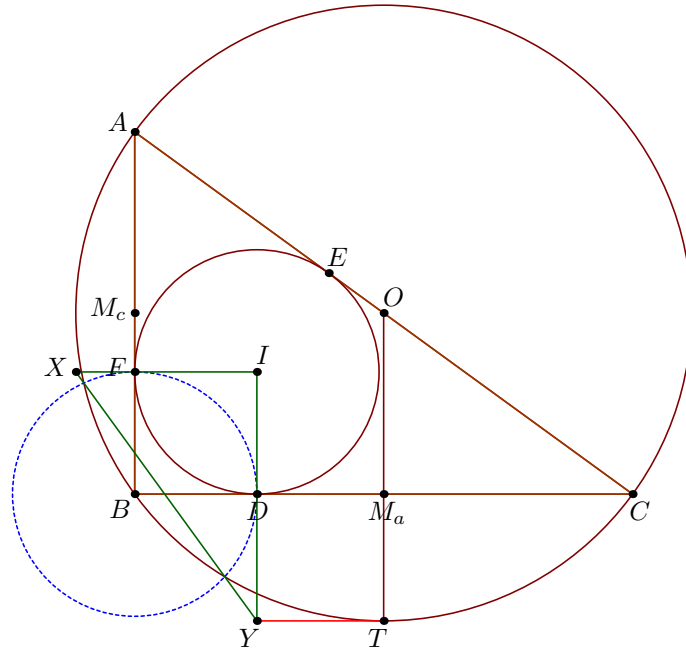
Let $\triangle ABC$ be a right triangle with right angle B with incenter I and intouch triangle $\triangle DEF$. Let the circle with center B containing D and F intersect the circumcircle at M and N . Let X and Y be the intersection point of \overline{MN} with \overline{ID} and \overline{IF} respectively. Show that X , Y , the midpoint of AB , and the midpoint of BC , are concyclic.

Video

<https://youtu.be/f4F7tDglpzA>

Solution

Let M_a, O, M_c be the midpoints of ABC . Then let T be the midpoint of minor arc \widehat{BC} .



Claim. $DM_a = M_aT$.

Proof. In the usual notation, $DM_a = BM_a - BD = \frac{a}{2} - \frac{b+c-a}{2} = \frac{b-c}{2}$, while $M_aT = OT - OM_a = \frac{b}{2} - \frac{c}{2}$. \square

Claim. $DYTM_a$ is a square.

Proof. Let Y' be the point such that DM_aTY' is a square. Then $Y'T^2 = Y'D^2$, while $Y'T$ is automatically tangent to (ABC) and $Y'D$ is automatically tangent to the circle centered at B with radius $BD = BF$. So $Y' = Y$. \square

Hence $DY = DM_a$, and Y is the reflection of M_a around the line \overline{DF} . The same is true for X , so XM_cM_aY is an isosceles trapezoid symmetric around line \overline{DF} .