Twitch 124.3

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Twitch Solves ISL

Episode 124

Problem

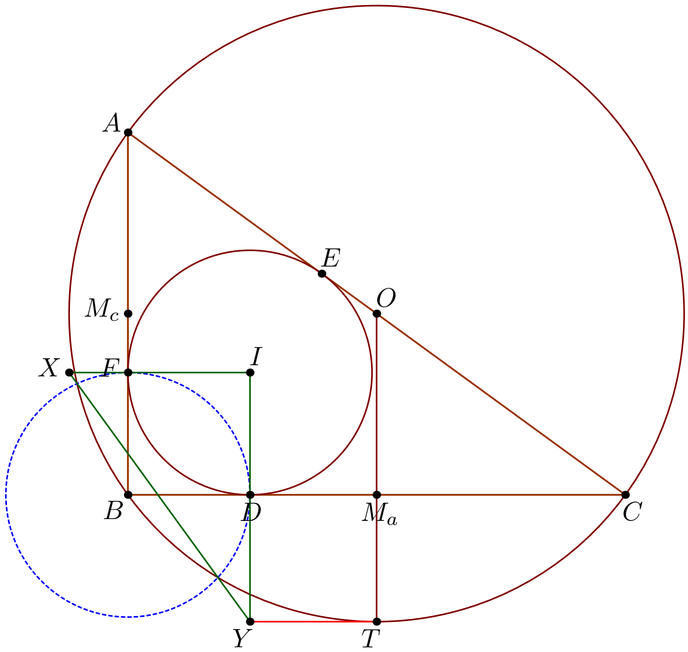
Let $\triangle ABC$ be a right triangle with right angle B with incenter I and intouch triangle $\triangle DEF$. Let the circle with center B containing D and F intersect the circumcircle at M and N. Let X and Y be the intersection point of $\overline{MN}$ with $\overline{ID}$ and $\overline{IF}$ respectively. Show that X, Y, the midpoint of AB, and the midpoint of BC, are concyclic.

Video

https://youtu.be/f4F7tDglpzA

Solution

Let M_a, O, M_c be the midpoints of ABC. Then let T be the midpoint of minor arc $\widehat{BC}$.



Claim. $DM_a = M_aT$.

Proof. In the usual notation, $DM_a = BM_a - BD = \frac{a}{2} - \frac{b+c-a}{2} = \frac{b-c}{2}$, while $M_aT = OT - OM_a = \frac{b}{2} - \frac{c}{2}$. □

Claim. $DYTM_a$ is a square.

Proof. Let Y' be the point such that DM_aTY' is a square. Then $Y'T^2 = Y'D^2$, while $Y'T$ is automatically tangent to (ABC) and $Y'D$ is automatically tangent to the circle centered at B with radius $BD = BF$. So $Y' = Y$. □

Hence $DY = DM_a$, and Y is the reflection of M_a around the line $\overline{DF}$. The same is true for X, so XM_cM_aY is an isosceles trapezoid symmetric around line $\overline{DF}$.