Longlist 1985/19

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Twitch Solves ISL

Episode 124

Problem

Solve over $\mathbb{R}$ the system of simultaneous equations

$$\begin{aligned} \sqrt{x} - \frac{1}{y} - 2w + 3z &= 1, \\ x + \frac{1}{y^2} - 4w^2 - 9z^2 &= 3, \\ x\sqrt{x} - \frac{1}{y^3} - 8w^3 + 27z^3 &= -5, \\ x^2 + \frac{1}{y^4} - 16w^4 - 81z^4 &= 15. \end{aligned}$$

Video

https://youtu.be/V_tTeGYaVAk

External Link

https://aops.com/community/p2014940

Solution

Let $a = \sqrt{x}$, $b = -1/y$, $c = 2w$, $d = -3z$.

$$\begin{aligned} a + b - c - d &= 1 \\ a^2 + b^2 - c^2 - d^2 &= 3 \\ a^3 + b^3 - c^3 - d^3 &= -5 \\ a^4 + b^4 - c^4 - d^4 &= 15. \end{aligned}$$

This condition is the same as saying

$$a^n + b^n + (-1)^n + (-1)^n = c^n + d^n + (-2)^n + 1^n \qquad n = 1, 2, 3, 4.$$

which is equivalent to saying the multiset $\{a, b, -1, -1\}$ is the same as the multiset $\{c, d, -2, 1\}$, (because Newton's formulas imply the polynomials with these roots have the same coefficients). Therefore, $\{a, b\} = \{-2, 1\}$ while $c = d = -1$.

Going back, with $a > 0$ this gives only one solution, which evidently works:

$$(x, y, w, z) = (1, 1/2, -1/2, 1/3).$$