Florida 2023B

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Twitch Solves ISL

Episode 124

Problem

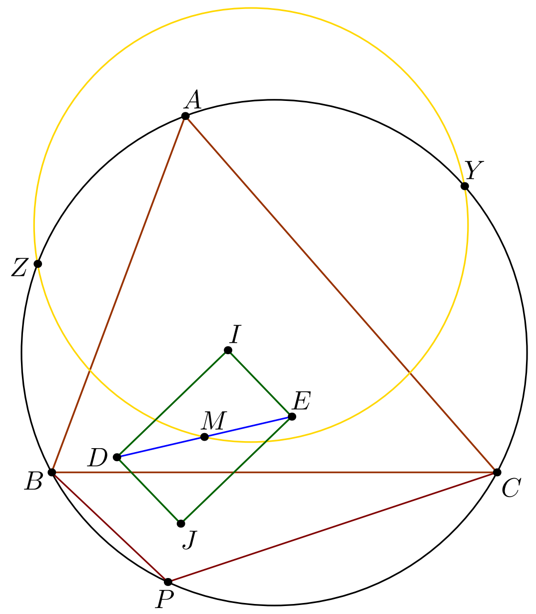
Given a fixed acute triangle, a variable point P lies on arc BC not containing A of the circumcircle of $\triangle ABC$. Let D and E be the incenters of ABP and ACP, respectively. As P varies on arc BC, show that the midpoint of $\overline{DE}$ lies on a fixed circle.

Video

https://youtu.be/NUyidWDwNls

Solution

Let I and J denote the incenters of triangles ABC and PBC.



We appeal to the following result, available here or here:

Theorem (Japanese theorem for cyclic quadrilaterals)**.** $DIEJ$ is a rectangle.

Now I is fixed, and J moves on a fixed circle (because $\angle BJC = 90^\circ + \frac{1}{2}\angle BPC$ is fixed). So the midpoint of $\overline{IJ}$ moves along on a circle, as needed.