EGMO 2023/2

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Twitch Solves ISL

Episode 124

Problem

We are given an acute triangle ABC. Let D be the point on its circumcircle such that AD is a diameter. Suppose that points K and L lie on segments AB and AC, respectively, and that DK and DL are tangent to circle AKL. Show that line KL passes through the orthocenter of triangle ABC.

Video

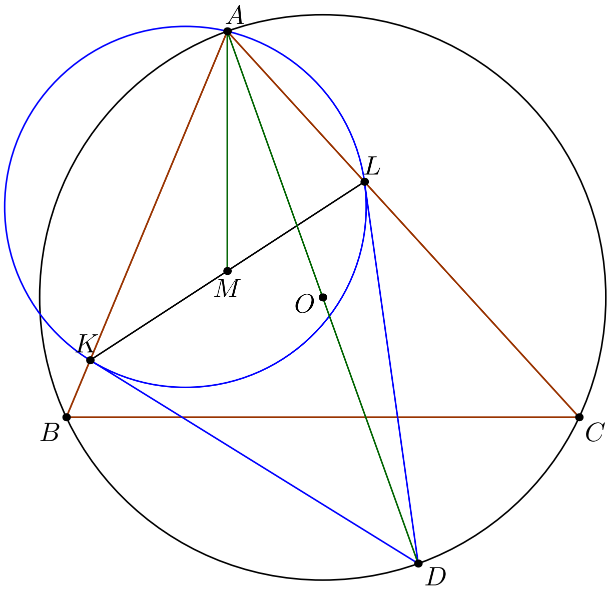
https://youtu.be/Gf_gS-scjoo

External Link

https://aops.com/community/p27523016

Solution

The problem consists of two disjoint "halves": the symmedian picture with $\triangle AKL$ and symmedian $\overline{AD}$, and the orthocenter picture with $\triangle ABC$ and diameter $\overline{AD}$. They are colored blue and red below, respectively.



Let M be the midpoint of $\overline{KL}$.

Claim (Blue part of the figure)**.** Lines $\overline{AM}$ and $\overline{AD}$ are isogonal to $\angle A$, and $AM = AD \cdot \cos A$.

Proof. First part follows from $\overline{AD}$ being the A-symmedian. It's also a standard fact that $AM \cdot AD = AK \cdot AL$ (e.g. by $\sqrt{bc}$ inversion); so it suffices to prove $AM^2 = AK \cdot AL \cos A$. This follows by Stewart's theorem to get AM together with the law of cosines $\cos A = \frac{AK^2+AL^2-KL^2}{2 \cdot AK \cdot AL}$. $\square$

Let H denote the orthocenter.

Claim (Red part of the figure)**.** Lines $\overline{AH}$ and $\overline{AO}$ are isogonal to $\angle A$, and $AH = AD \cdot \cos A$.

Proof. Both parts are well-known. $\square$

The two claims together imply $H = M$ and we're done.