CEMC Euclid 2023/10C

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Twitch Solves ISL

Episode 124

Problem

Define
$$f(n) = \left\lfloor \frac{n}{1^2 + 1} \right\rfloor + \left\lfloor \frac{2n}{2^2 + 1} \right\rfloor + \left\lfloor \frac{3n}{3^2 + 1} \right\rfloor + \cdots$$
for each positive integer n. Suppose m is odd and $f(m+1) - f(m) = 2$. Show that m is prime.

Video

External Link

Solution

Notice that for any positive integers n and k, we have

$$\left\lfloor \frac{kn+k}{k^2+1} \right\rfloor - \left\lfloor \frac{kn}{k^2+1} \right\rfloor = \begin{cases} 1 & n \equiv k, 2k, \ldots, k^2 \pmod{k^2+1} \\ 0 & \text{otherwise.} \end{cases}$$

Fix m and let a_k be the term above for brevity. This means that

$$f(m+1) - f(m) = a_1 + a_2 + \ldots.$$

Claim. If $m > 1$ is odd and composite, then $a_1 + a_2 + \ldots$ is at least 3.

Proof. Notice that $a_1 = 1$, $a_m = 1$, and if d is any proper divisor of m which is greater than or equal to $\sqrt{m}$ (so that $d^2 + 1 > m$), we have $a_d = 1$ as well. $\square$

Finally, $f(2) - f(1) = 1$. So if m is odd and $f(m+1) - f(m) = 2$, it follows m is prime.