CEMC Euclid 2023/10C Evan Chen

TWITCH SOLVES ISL

Episode 124

Problem

Define

$$f(n) = \left\lfloor \frac{n}{1^2 + 1} \right\rfloor + \left\lfloor \frac{2n}{2^2 + 1} \right\rfloor + \left\lfloor \frac{3n}{3^2 + 1} \right\rfloor + \cdots$$

for each positive integer n. Suppose m is odd and f(m+1) - f(m) = 2. Show that m is prime.

Video

https://youtu.be/soMY1qyNiNI

External Link

https://aops.com/community/p27446040

Solution

Notice that for any positive integers n and k, we have

$$\left\lfloor \frac{kn+k}{k^2+1} \right\rfloor - \left\lfloor \frac{kn}{k^2+1} \right\rfloor = \begin{cases} 1 & n \equiv k, 2k, \dots, k^2 \pmod{k^2+1} \\ 0 & \text{otherwise.} \end{cases}$$

Fix m and let a_k be the term above for brevity. This means that

$$f(m+1) - f(m) = a_1 + a_2 + \dots$$

Claim. If m > 1 is odd and composite, then $a_1 + a_2 + \ldots$ is at least 3.

Proof. Notice that $a_1 = 1$, $a_m = 1$, and if d is any proper divisor of m which is greater than or equal to \sqrt{m} (so that $d^2 + 1 > m$), we have $a_d = 1$ as well.

Finally, f(2) - f(1) = 1. So if m is odd and f(m+1) - f(m) = 2, it follows m is prime.