

Twitch 123.4

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TWITCH SOLVES ISL

Episode 123

Problem

Find the real numbers x and y that minimize the sum

$$\sqrt{2(x^2 + y^2)} + \sqrt{2(x^2 + y^2) + 2(x - y) + 1} + \sqrt{2(x^2 + y^2) + 2(x + y) + 1}.$$

Video

<https://youtu.be/Qvq4pTg-C8o>

Solution

Let S be sum. Then

$$\begin{aligned}\frac{S}{\sqrt{2}} &= \sqrt{x^2 + y^2} \\ &+ \sqrt{(x + 1/2)^2 + (y - 1/2)^2} \\ &+ \sqrt{(x + 1/2)^2 + (y + 1/2)^2}\end{aligned}$$

Let

$$\begin{aligned}A &= (0, 0) \\ B &= (-1/2, 1/2) \\ C &= (-1/2, -1/2).\end{aligned}$$

Want to minimize $PA + PB + PC$ where $P \in \mathbb{R}^2$, so take P to be the Fermat point:

$$P = \left(-\frac{1}{2} + \frac{1}{2\sqrt{3}}, 0\right).$$