EGMO 2023/6

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Twitch Solves ISL

Episode 123

Problem

Let ABC be a triangle with circumcircle Ω. Let S_b and S_c respectively denote the midpoints of the arcs AC and AB that do not contain the third vertex. Let N_a denote the midpoint of arc BAC (the arc BC including A). Let I be the incenter of ABC. Let ω_b be the circle that is tangent to AB and internally tangent to Ω at S_b, and let ω_c be the circle that is tangent to AC and internally tangent to Ω at S_c. Show that the line IN_a, and the lines through the intersections of ω_b and ω_c, meet on Ω.

Video

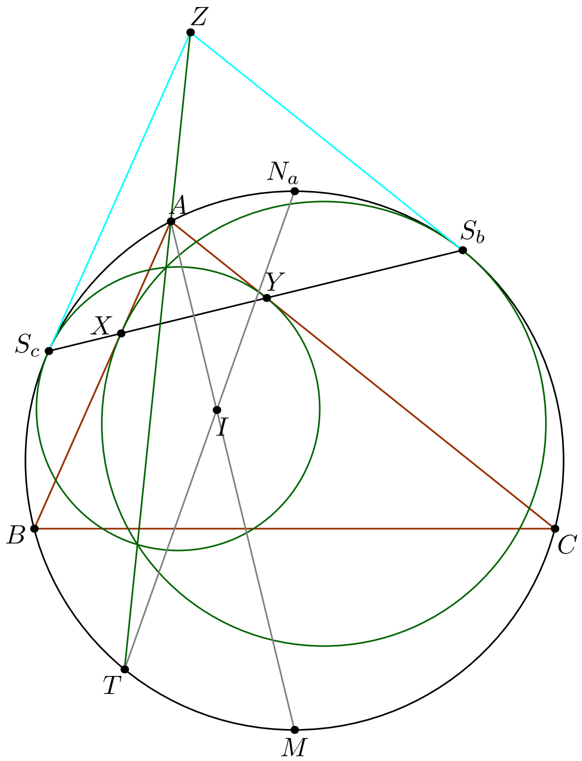
https://youtu.be/ZmG11r8tQNI

External Link

https://aops.com/community/p27522960

Solution

Let $Z = \overline{S_bS_b} \cap \overline{S_cS_c}$, $X = \overline{AB} \cap \overline{S_BS_c}$ (which lies on ω_b by shooting lemma), and $Y = \overline{AC} \cap \overline{S_BS_c}$ (which lies on ω_c similarly).



Claim. Point A lies on the radical axis of ω_b and ω_c.

Proof. It's well known $\overline{S_bS_c}$ is the perpendicular bisector of $\overline{AI}$, so we get $AX = AY$ and hence $AX^2 = AY^2$. □

Claim. Point Z lies on the radical axis of ω_b and ω_c.

Proof. It's the radical center of ω_b, ω_c, Ω. □

Claim. Quadrilateral AS_bTS_c is harmonic.

Proof. $(AT; S_bS_c) \stackrel{I}{=} (MN_a; BC) = -1$. □

The third claim implies A, T, Z are collinear, solving the problem.