EGMO 2023/3

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Twitch Solves ISL

Episode 123

Problem

Let k be a fixed positive integer. Lexi has a dictionary $\mathbb{D}$ consisting of some k-letter strings containing only the letters A and B. Lexi would like to write either the letter A or the letter B in each cell of a $k \times k$ grid so that each column contains a string from $\mathbb{D}$ when read from top-to-bottom and each row contains a string from $\mathbb{D}$ when read from left-to-right.

What is the smallest integer m such that if $\mathbb{D}$ contains at least m different strings, then Lexi can fill her grid in this manner, no matter what strings are in $\mathbb{D}$?

Video

https://youtu.be/PcPjYpkwY_g

External Link

https://aops.com/community/p27522967

Solution

Answer: $|\mathbb{D}| \geq 2^{k-1}$ is sufficient to fill the grid.

Counterexample where $|\mathbb{D}| = 2^{k-1} - 1$. Let $\mathbb{D}$ be all the words that start with A other than the all-A string. There can't be any construction because the first row would contain a B, but no words in $\mathbb{D}$ start with B.

Sufficiency proof when $|\mathbb{D}| \geq 2^{k-1}$. If $\mathbb{D}$ contains either the all-A or all-B string, we're done.

Otherwise, pair the remaining $2^k - 2$ possible strings into $2^{k-1} - 1$ pairs where we pair two strings if they have no overlapping letters in the same position (i.e. they are *opposites*, like $ABBAA$ and $BAABB$). Because $|\mathbb{D}| > 2^{k-1} - 1$, it follows $\mathbb{D}$ has a pair of such opposites in it.

Then it's possible to fill the grid with just those two words, e.g. $ABBAA$ gives the grid

$$\begin{bmatrix} A & B & B & A & A \\ B & A & A & B & B \\ B & A & A & B & B \\ A & B & B & A & A \\ A & B & B & A & A \end{bmatrix}.$$