China 2021/6

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Twitch Solves ISL

Episode 123

Problem

Solve over positive integers the functional equation

$$f(f(x) + y) \mid x + f(y).$$

Video

https://youtu.be/orvTuytr6uk

External Link

https://aops.com/community/p19116098

Solution

There are three families of solutions:

- f is the identity function;
- $f(n) = 1$ for $n \geq 2$, where $f(1)$ is any positive integer.
- $f(n) = 2$ for n even, $f(n) = 1$ for odd $n \geq 3$, and $f(1)$ is any odd positive integer.

The verification is easy, so we prove these are the only solution. The proof is split into two main cases.

Case where f is not injective. Let

$$T = \min_{f(a)=f(b), a<b} (b - a).$$

Then all outputs of f are eventually divisors of T, as

$$f(y + f(a)) = f(y + f(b)) \mid \gcd(f(y) + a, f(y) + b) \mid b - a \qquad \forall y > 0.$$

Claim. We have $T \leq 2$.

Proof. If $T > 2$, then look any T consecutive outputs. They were supposed to be distinct divisors of T, impossible for $T > 2$. □

If $T = 1$, then for any $n \geq 2$, let $y = n - 1$ and let x be a large integer. Then we have $f(n) \mid x + f(n - 1)$ for all large x, forcing $f(n) + 1$.

If $T = 2$, then it follows f must alternate between 1 and 2 eventually. Again, $f(n) \leq 2$ follows for $n \geq 2$ as in the previous case, by letting $y = n - 1$ and x be large with $f(x) = 1$. We can then quickly verify that only the situation where $f(\text{even}) = \text{even}$ bears solutions, the ones we claimed earlier.

Case where f is not injective. Let $c = f(1)$. Taking $x = 1$ then gives

$$f(y + c) \leq f(y) + 1.$$

This implies

$$f(n) \leq \frac{1}{c}n + \max\left\{f(1), \ldots, f(c)\right\}.$$

For f to be injective, we must then have $c = 1$. Finally, $f(y + 1) \leq f(y) + 1$ together with f injective then forces f to be the identity function, by induction.