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Twitch Solves ISL

Episode 122

Problem

Find the smallest real number C, such that for any positive integers $x \neq y$ holds the following:

$$\min\left(\left\{\sqrt{x^2+2y}\right\},\left\{\sqrt{y^2+2x}\right\}\right) < C$$

where $\{\dots\}$ denotes the fractional part.

Video

https://youtu.be/y8k9qrrG-Qc

External Link

https://aops.com/community/p27437949

Solution

The answer is $C = \frac{\sqrt{5}-1}{2}$.

First, we prove this works. WLOG, $x < y$. Then observe two cases:

- Whenever $x < Cy$, we have
$$y < \sqrt{y^2 + 2x} < y + C.$$
- Also, whenever $Cy < x < y$ we have
$$x + 1 < \sqrt{x^2 + 2y} < x + (C + 1).$$

Thus, in each case, one of the fractional parts is less than C.

Conversely, for the construction, we really just need $x/y \approx C$. Consider any increasing sequences x_n, y_n of positive integers with $\lim_{n\to\infty} \frac{x_n}{y_n} = C$. (For example, one could take $y_n = 10^n$ and $x_n = \lfloor 10^n C \rfloor$, or take y_n and x_n to be consecutive Lucas/Fibonacci numbers.) For brevity, abbreviate x_n to x and y_n to y. Then

$$\begin{aligned}\sqrt{y^2 + 2x} - y = \frac{2x}{\sqrt{y^2 + 2x} + y} &= \frac{2}{\sqrt{\left(\frac{y}{x}\right)^2 + \frac{2}{x}} + \frac{y}{x}} \\ &\xrightarrow{n\to\infty} \frac{2}{\sqrt{(1/C)^2} + (1/C)} = C.\end{aligned}$$

and similarly

$$\begin{aligned}\sqrt{x^2 + 2y} - (x + 1) = \frac{2y - 2x - 1}{\sqrt{x^2 + 2y} + x + 1} &= \frac{2 - \frac{1}{y-x}}{\sqrt{\left(\frac{x}{y-x}\right)^2 + \frac{2}{y-x}} + \frac{x}{y-x} + \frac{1}{y-x}} \\ &\xrightarrow{n\to\infty} \frac{2}{\sqrt{\left(\frac{C}{1-C}\right)^2} + \frac{C}{1-C}} = C.\end{aligned}$$