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Twitch Solves ISL

Episode 122

Problem

Find all functions f taking integer-lattice vectors in $\mathbb{Z}^2$ to $\mathbb{R}$ such that $f(\mathbf{a}+\mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b})$ whenever $\mathbf{a} \perp \mathbf{b}$.

Video

https://youtu.be/2mn3De0yc8A

External Link

https://aops.com/community/p27461531

Solution

Abbreviate $f(\langle x, y\rangle)$ to just $f(x, y)$. Note that the set of solutions is obviously an $\mathbb{R}$-vector space. Consider the four solutions:

$$\begin{aligned} f_1(x,y) &\coloneqq x \\ f_2(x,y) &\coloneqq y \\ f_3(x,y) &\coloneqq x^2 + y^2 \\ f_4(x,y) &\coloneqq (x\%2) - (y\%2). \end{aligned}$$

(Verification left as exercise.) They are linearly independent, so any linear combination of this works, and we have found a 4-dimensional space of solutions.

We now show conversely that the dimension of the vector space of solutions is at most 4. To begin, first note that

$$f(x,y) = f(x,0) + f(0,y).$$

In particular, $f(0,0) = 0$. Next, note that for all $n \geq 1$

$$\begin{aligned} f(n,0) + f(0,-1) + f(1,0) + f(0,n) &= f(n,-1) + f(1,n) \\ &= f(n+1, n-1) \\ &= f(n+1,0) + f(0,n-1) \\ \implies f(n+1,0) &= f(n,0) + f(0,-1) + f(1,0) + f(0,n) - f(0,n-1) \end{aligned}$$

By writing the analogous equations for $f(-(n+1),0)$, $f(0,n+1)$ and $f(0,-(n+1))$, it follows that all the values of f are determined by these recursions as soon as the "initial" values of $f(1,0)$, $f(-1,0)$, $f(0,1)$, $f(0,-1)$ are chosen. Ergo, the space of valid f is at most four-dimensional, as needed.