China 2005/2

Evan Chen

Twitch Solves ISL

Episode 122

Problem

Over all $a, b, c, x, y, z > 0$ such that $bz + cy = a$, $az + cx = b$, $ay + bx = c$, find the smallest possible value of

$$\frac{x^2}{1+x} + \frac{y^2}{1+y} + \frac{z^2}{1+z}.$$

Video

https://youtu.be/kFaKtqOXGk0

External Link

https://aops.com/community/p3592156

Solution

The answer is $1/2$, achieved at $a = b = c$ and $x = y = z = 1/2$.

To prove the inequality, note that if we solve for x, y, z in terms of a, b, c we get

$$\begin{aligned} x &= \frac{-a^2+b^2+c^2}{2bc} \\ y &= \frac{a^2-b^2+c^2}{2ac} \\ z &= \frac{a^2+b^2-c^2}{2ab}. \end{aligned}$$

If $x, y, z > 0$, then it follows that a, b, c are the sides of an acute triangle. Then, by the law of cosines, we actually have

$$x = \cos A, \qquad y = \cos B, \qquad z = \cos C.$$

Now, let $s := x + y + z$.

Claim. We have $s \ge 3/2$.

Proof. Since $0 < A, B, C < \frac{1}{2}\pi$, it follows by Jensen on cos. $\square$

Then by Cauchy-Schwarz

$$\frac{x^2}{1+x} + \frac{y^2}{1+y} + \frac{z^2}{1+z} \ge \frac{(x+y+z)^2}{3+x+y+z} = \frac{s^2}{3+s} \ge \frac{1}{2}$$

since $\frac{s^2}{3+s} \ge \frac{1}{2} \iff (2s-3)(s+1) \ge 0$.