RMM 2008/4

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Twitch Solves ISL

Episode 121

Problem

Consider a square of sidelength n and $(n+1)^2$ interior points. Prove that we can choose 3 of these points so that they determine a triangle (possibly degenerate) of area at most $\frac{1}{2}$.

Video

https://youtu.be/79W3dNyiYWw

External Link

https://aops.com/community/p1030132

Solution

Assume no 3 points are collinear (else they are area 0). Suppose the convex hull of our polygon has $k \geq 3$ points. We consider two cases.

First case: $k \leq 4n$. If $k \leq 4n$, we use the following algorithm: draw the entire convex hull, and then repeatedly draw non-intersecting segments between pairs of points. The end result is a division of the k-gon into triangles.

By double-counting interior angles, the number of triangles is given exactly by

$$\frac{360^\circ \cdot \left((n+1)^2 - k\right) + 180^\circ \cdot (k-2)}{180^\circ} = 2n^2 + 4n - k \geq 2n^2.$$

The sum of the areas of the triangles is at most n^2 (the total area of the square), so we're done.

Second case: $k \geq 4n$. We will show that we can take some three consecutive vertices of the convex hull.

Let $a_1, \dots, a_k$ be the side lengths of the convex hull in order. A convex polygon contained inside a square of side length n has perimeter at most $4n$, so we have $a_1 + \dots + a_k \leq 4n$. Using the lemma with $k \geq 4n$, there should be an index i with $a_i + a_{i+1} \leq 2$. It follows that $a_i a_{i+1} \leq 1$ for some index i by AM-GM. The two associated sides then carve out a triangle of area at most $\frac{1}{2}$.