# Longlist 1974/2

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Twitch Solves ISL

Episode 121

## Problem

Let $u_n$ be the Fibonacci sequence, i.e., $u_0 = 0$, $u_1 = 1$, $u_n = u_{n-1} + u_{n-2}$ for $n > 1$. Prove that there exist infinitely many prime numbers $p$ that divide $u_{p-1}$.

## Video

https://youtu.be/QS7jBXIVf8s

## External Link

https://aops.com/community/p2136111

## Solution

Let $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$. We have Binet's formula:

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

In fact, if $p \equiv 1 \pmod 5$, then by quadratic reciprocity, 5 is a quadratic residue, so $\alpha$ and $\beta$ can be viewed as elements of $\mathbb{F}_p$, and Fermat's theorem applies to give $\alpha^{p-1} \equiv \beta^{p-1} \equiv 1 \pmod p$, as desired.

**Remark.** In fact, if $p \equiv \pm 2 \pmod 5$ and $p > 2$ then $\alpha^p = \beta$ and $\beta^p = \alpha$, so in this case

$$u_{p-1} \equiv \frac{\frac{\beta}{\alpha} - \frac{\alpha}{\beta}}{\alpha - \beta} = -\frac{\alpha+\beta}{\alpha\beta} = -\frac{-1}{1} = 1 \pmod p.$$