BAMO 2019/3

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Twitch Solves ISL

Episode 121

Problem

In $\triangle ABC$, mark points A', B', and C' on segments $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively, so that $\overline{AA'}$ is an altitude, $\overline{BB'}$ is a median, and $\overline{CC'}$ is an angle bisector. Prove that if $\triangle A'B'C'$ is equilateral if and only if $\triangle ABC$ is equilateral.

Video

https://youtu.be/4VQxYmE2tEQ

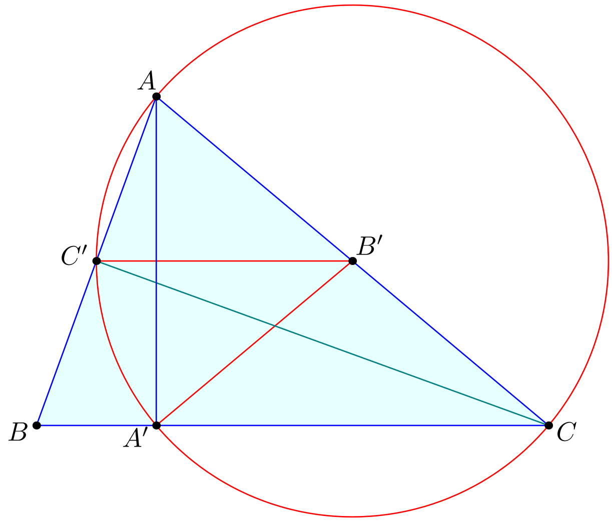
External Link

https://aops.com/community/p13025889

Solution

If $\triangle ABC$ is equilateral, it's obvious because A', B', C' are the midpoints.

Conversely, consider any triangle. Note that A' unconditionally lies on the circle with diameter $\overline{AC}$ centered at B', meaning $A'B' = B'A = B'C$.



So if $\triangle A'B'C'$ was equilateral, then $C'B' = B'A'$, then C' would lie on this circle as well. This means $\angle AC'C = 90^\circ$. As C' was the foot of the bisector, it follows $AC = BC$.

To finish, note that

$$60^\circ = \angle C'B'A' = 2\angle C'CA' = \angle ACB.$$