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Evan Chen

Twitch Solves ISL

Episode 120

Problem

Quadrilateral $ABCD$ is such that $2\angle A = \angle B$ and $2\angle D = \angle C$. If $AB = 3$, $BC = 4$, and $CD = 5$, find the area of quadrilateral $ABCD$.

Video

https://youtu.be/PBSOcmLosrw

External Link

https://aops.com/community/p24130067

Solution

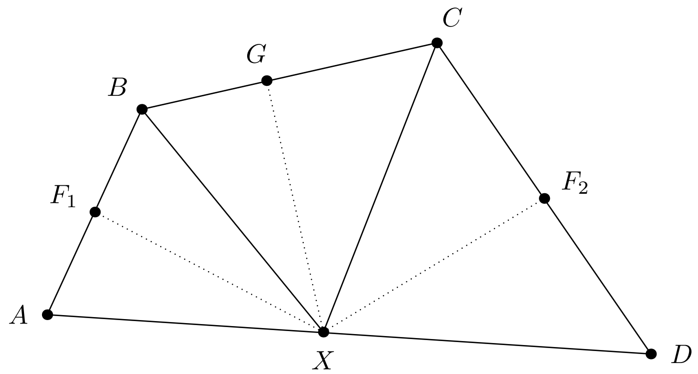
This problem and solution were contributed by Benny Wang.

Let E be the intersection of the angle bisector of $\angle ABC$ with $\overline{AD}$; let F be the intersection of the angle bisector of $\angle BCD$ with $\overline{AD}$.

Claim. We have $E = F$.

Proof. Let the perpendiculars from E to $\overline{AB}$ and $\overline{BC}$ be E_1 and E_2. Note that due to $\triangle ABE$ isosceles with $AE = BE$ (using the angle condition), it follows that $E_1B = \frac{AB}{2} = 1.5$. Then, since we defined $\overline{BE}$ as the bisector of $\angle ABC$, we have that $E_2B = E_1B = 1.5$.

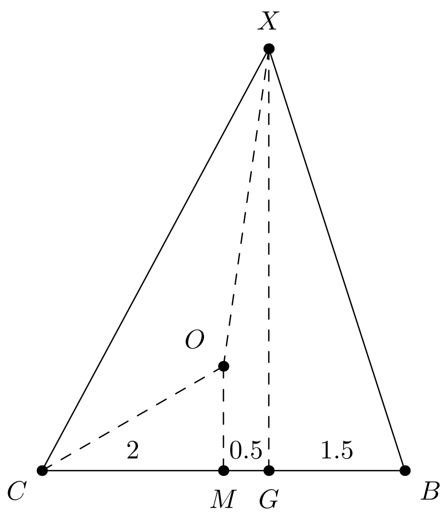
Similarly, if we define F_1 and F_2 as the perpendiculars from F to $\overline{CD}$ and $\overline{BC}$, we have that $F_2C = F_1C = \frac{CD}{2} = 2.5$. Note that $BE_2 + F_2C = 4 = BC$, implying that $E_2 = F_2$, implying that $E = F$, as desired. $\square$



Now we compute a bunch. Let G be the point $E_2 = F_2$, and let X be the point $E = F$.

Consider $\triangle BXC$. Then G is defined as the foot of the altitude from X to $\overline{BC}$; we know $BG = 1.5$ and $GC = 2.5$. We also know that $\angle BXC = 60^\circ$ by some straightforward angle chasing. This is enough information to solve for $\overline{BXC}$; in particular, we desire the length of XG; then, since X is equidistant from $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$, we could compute the area of $ABCD$ to be $\frac{(AB+BC+CD)\times XG}{2}$.

To find XG is a straightforward geometry problem now. There are several ways we could go about this. Here is one: let M be the midpoint of $\overline{BC}$, and let O be the circumcenter of $\triangle BXC$.



Note that $\angle BOC = 2\angle BXC = 120^\circ$, so $\triangle BMO$ and $\triangle CMO$ are 30-60-90 triangles. Therefore, $MO = \frac{2}{\sqrt{3}}$ and $BO = OC = OX = \frac{4}{\sqrt{3}}$. Now, note that

$$\begin{aligned} GX &= \sqrt{XO^2 - GM^2} + MO \\ &= \sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - 0.5^2} + \frac{2}{\sqrt{3}} \\ &= \sqrt{\frac{61}{12}} + \frac{4}{\sqrt{12}} \\ &= \frac{4+\sqrt{61}}{\sqrt{12}}. \end{aligned}$$

Therefore,

$$[ABCD] = \frac{XG \cdot (AB + BC + CD)}{2} = \frac{4+\sqrt{61}}{2\sqrt{12}} \cdot 12 = \boxed{4\sqrt{3} + \sqrt{183}}.$$