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Twitch Solves ISL

Episode 119

Problem

The graph of $y = f(x)$ for some cubic polynomial $f \in \mathbb{R}[x]$ meets the x-axis at points A, B, and $(20, 0)$. A line ℓ of slope $\frac{1}{13}$ meets the cubic at points P, Q, and $(23, u)$. Given that A, B, P, and Q are distinct and concyclic, find u.

Video

https://youtu.be/8ij1wFg-SjU

External Link

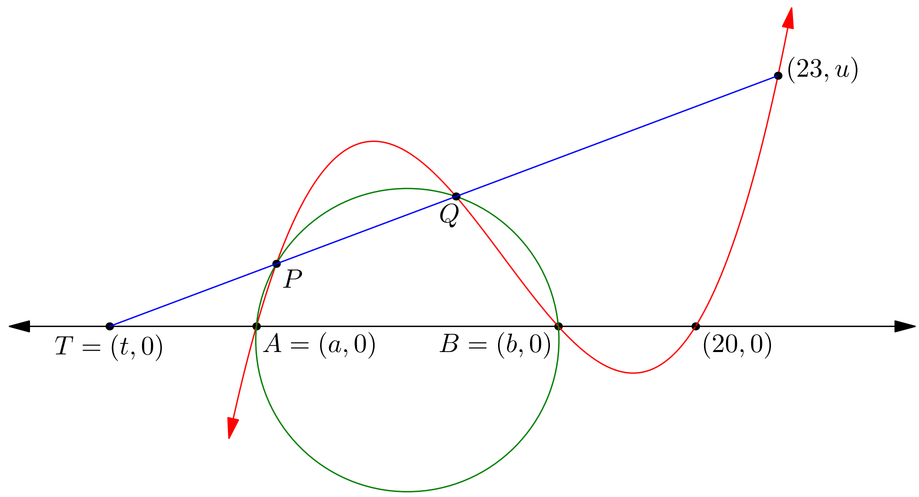
https://aops.com/community/p26889787

Solution

Set $A = (a, 0)$, $B = (b, 0)$, $P = (p, f(p))$, $Q = (q, f(q))$. Let line AB meet the x-axis at $T = (t, 0)$. Finally, let $f(x) = k(x-a)(x-b)(x-20)$ for $k \in \mathbb{R}$.

By power of a point, we have

$$\frac{\sqrt{170}}{13}(t-p) \cdot \frac{\sqrt{170}}{13}(t-q) = TP \cdot TQ = TA \cdot TB = (t-a)(t-b).$$



Meanwhile, the equation of $\overline{TPQ} \cap \operatorname{Graph}(f)$ is given by

$$k(x-a)(x-b)(x-20) - \frac{x-t}{13} \equiv k(x-23)(x-p)(x-q).$$

since the right-hand side is a cubic polynomial with roots at 23, p, q. Plugging in $x = t$ to this now gives

$$\begin{aligned} k(t-a)(t-b)(t-20) &= k(t-23)(t-p)(t-q) \\ \implies \frac{170}{169}(t-20) &= (t-23) \\ \implies t &= -487 \end{aligned}$$

Finally, $u = \frac{23-t}{13} = \frac{510}{13}$.