bissue 2022/13 Evan Chen

TWITCH SOLVES ISL

Episode 119

Problem

The graph of y = f(x) for some cubic polynomial $f \in \mathbb{R}[x]$ meets the x-axis at points A, B, and (20,0). A line ℓ of slope $\frac{1}{13}$ meets the cubic at points P, Q, and (23, u). Given that A, B, P, and Q are distinct and concyclic, find u.

Video

https://youtu.be/8ij1wFg-SjU

External Link

https://aops.com/community/p26889787

Solution

Set A = (a, 0), B = (b, 0), P = (p, f(p)), Q = (q, f(q)). Let line AB meet the x-axis at T = (t, 0). Finally, let f(x) = k(x - a)(x - b)(x - 20) for $k \in \mathbb{R}$.

By power of a point, we have



Meanwhile, the equation of $\overline{TPQ} \cap \operatorname{Graph}(f)$ is given by

$$k(x-a)(x-b)(x-20) - \frac{x-t}{13} \equiv k(x-23)(x-p)(x-q)$$

since the right-hand side is a cubic polynomial with roots at 23, p, q. Plugging in x = t to this now gives

$$k(t-a)(t-b)(t-20) = k(t-23)(t-p)(t-q)$$

$$\implies \frac{170}{169}(t-20) = (t-23)$$

$$\implies t = -487$$

Finally, $u = \frac{23-t}{13} = \frac{510}{13}$.