Twitch 119.2

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Twitch Solves ISL

Episode 119

Problem

Let n be a positive integer. There are $2n$ kids in a circle each given 2 cards, where the $4n$ cards are numbered uniquely from 1 to $4n$. Every second, each child instantaneously passes their larger-valued card to the child on their right and their smaller-valued card to the child on the left. The game ends when a child has two cards of consecutive value. Show that if the game does not end within the first $n - 1$ seconds, then the game will never end.

Video

https://youtu.be/wwZUgX_R3iQ

Solution

Change the rules such that a child keeps their largest card and passes the smaller card two kids to their left. Hence the largest cards stay in place. Moreover, only kids of the same parity around the circle ever interact, so we may restrict attention to the n even-numbered kids, say. Let $C_1 < C_2 < \cdots < C_{2n}$ denote their $2n$ cards in order.

We color the cards C_n, C_{n+1}, ..., C_{2n} *green*. The green cards have the property that:

- A child with two green cards will pass one of them;

- A child with one green cards passes the non-green one;

- A child with no green cards passes either one.

We are going to use the following lemma now:

Lemma (Raney)**.** In a sequence of integers $(x_1, \ldots, x_n)$ with sum 1, there exists a unique cyclic shift with all partial sums strictly positive.

Apply the lemma to the sequence of integers which is $\{-1, 0, 1\}$ at a kid according to whether they have zero, one, or two green cards to start. Then label the kids around the circle K_1, K_2, ..., K_n according to the unique cyclic shift given by the lemma.

Claim. We have:

- At the end of the process, the cards C_{n+1}, ..., C_{2n} are held by different kids.

- Right after the $(n-1)$st second, C_n and C_{n+1} are held by kid K_n.

Proof. To be written later (it does not require any idea and sort of follows by the construction, but it requires a lot of words to explain well). $\qquad\square$

So there are two cases:

- If C_n and C_{n+1} had non-consecutive numbers, then no kid will ever hold cards of consecutive numbers again (since they always have one card at least C_{n+1} and one card at most C_n).

- If C_n and C_{n+1} are consecutive the game ended already after at most $n-1$ seconds.