Twitch 118.1

Evan Chen

Twitch Solves ISL

Episode 118

Problem

Prove that for real numbers $a \geq 0$,

$$\sqrt{a^2 + 5 - \sqrt{a^2 + 6a}} + \sqrt{a^2 + 7a + 8} \geq 5.$$

Video

https://youtu.be/W5EgJeHEIec

Solution

Two approaches.

First solution (via Cauchy). Rearranging:

$$\begin{aligned}\iff a^2+5-\sqrt{a^2+6a} &\ge \left(5-\sqrt{a^2+7a+8}\right)^2 \\ &= 25-10\sqrt{a^2+7a+8}+(a^2+7a+8) \\ \iff 10\sqrt{a^2+7a+8} &\ge 28+7a+\sqrt{a^2+6a}.\end{aligned}$$

The latter claim follows by Cauchy-Schwarz in the form

$$(49+1)\left((a+4)^2+(a^2+6a)\right) \ge \left(7(a+4)+\sqrt{a^2+6a}\right)^2.$$

Second solution (geo). Let $b=\sqrt{a^2+6a}$; then this reads

$$\sqrt{(a-3)^2+(b-1)^2}+\sqrt{(a+4)^2+b^2} \ge \sqrt{50}$$

which follows from a geometric argument.