

Twitch 118.1

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TWITCH SOLVES ISL

Episode 118

Problem

Prove that for real numbers $a \geq 0$,

$$\sqrt{a^2 + 5} - \sqrt{a^2 + 6a} + \sqrt{a^2 + 7a + 8} \geq 5.$$

Video

<https://youtu.be/W5EgJeHEIec>

Solution

Two approaches.

First solution (via Cauchy) Rearranging:

$$\begin{aligned} \Leftrightarrow a^2 + 5 - \sqrt{a^2 + 6a} &\geq \left(5 - \sqrt{a^2 + 7a + 8}\right)^2 \\ &= 25 - 10\sqrt{a^2 + 7a + 8} + (a^2 + 7a + 8) \\ \Leftrightarrow 10\sqrt{a^2 + 7a + 8} &\geq 28 + 7a + \sqrt{a^2 + 6a}. \end{aligned}$$

The latter claim follows by Cauchy-Schwarz in the form

$$(49 + 1) \left((a + 4)^2 + (a^2 + 6a) \right) \geq \left(7(a + 4) + \sqrt{a^2 + 6a} \right)^2.$$

Second solution (geo) Let $b = \sqrt{a^2 + 6a}$; then this reads

$$\sqrt{(a - 3)^2 + (b - 1)^2} + \sqrt{(a + 4)^2 + b^2} \geq \sqrt{50}$$

which follows from a geometric argument.