IMO 1987/5

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Twitch Solves ISL

Episode 117

Problem

Is it possible to put 1987 points in the Euclidean plane such that the distance between each pair of points is irrational and each three points determine a non-degenerate triangle with rational area?

Video

https://youtu.be/MQizW3_nrZE

External Link

https://aops.com/community/p366548

Solution

Note that any triangle with vertices which are lattice points has rational area. So we'd be done if we could prove:

Claim. Given a finite set S of lattice points, we can add one lattice point P such that P is irrational distance from every point in S.

Proof. Suppose S lies inside $[1,n] \times [1,n]$. Let $a^2 + b^2 = c^2$ by a primitive Pythagorean triple with the additional property that $a/b \neq x/y$ for any $(x,y) \in S$; this is possible there are infinitely many primitive Pythagorean triples.

Consider large R; I claim that $P = (aR, bR)$ is okay if R is sufficiently large. The squared distance from P to a point (u,v) is

$$D = (aR-u)^2 + (bR-v)^2 = (cR)^2 - 2(au+bv)R + (u^2+v^2)$$

So for this to be square, it must be inside the set

$$D \in \{(5R-(a+b)n)^2, (5R-(a+b)n+1)^2, \dots, (5R+2n)^2\}.$$

In other words, if R is large enough, then D is not a square unless D is actually identically the square of a polynomial in R. For that to be the case, we would need

$$D = \left(cR - \sqrt{u^2+v^2}\right)^2$$

which would only happen if

$$c\sqrt{u^2+v^2} = au+bv \implies 0 = c^2(u^2+v^2) - (au+bv)^2 = (bu-av)^2$$

which doesn't happen because we assured $a/b \neq u/v$.

Hence for each $(u,v) \in S$, there exists a constant $C_{u,v}$ for which $R > C_{u,v}$ guarantees P is irrational distance from (u,v). Take the max of these finitely many constants to finish. □