IMO 1997/3

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Twitch Solves ISL

Episode 116

Problem

Let x_1, x_2, $\dots$, x_n be real numbers satisfying the conditions:

$$|x_1 + x_2 + \dots + x_n| = 1$$
$$|x_i| \le \frac{n+1}{2} \qquad \text{for } i = 1, 2, \dots, n$$

Show that there exists a permutation y_1, y_2, $\dots$, y_n of x_1, x_2, $\dots$, x_n such that

$$|y_1 + 2y_2 + \dots + ny_n| \le \frac{n+1}{2}.$$

Video

https://youtu.be/zY4Uxk7i2Gg

External Link

https://aops.com/community/p356706

Solution

WLOG $\sum x_i = 1$ (by negating x_i) and $x_1 \le x_2 \le \dots \le x_n$. Notice that

- The largest possible value of the sum in question is

$$A = x_1 + 2x_2 + 3x_3 + \dots + nx_n.$$

  while the smallest value is

$$B = nx_1 + (n-1)x_2 + \dots + x_n.$$

- Meanwhile, the *average* value across all permutations is

$$1 \cdot \frac{n+1}{2} + 2 \cdot \frac{n+1}{2} + \dots + n \cdot \frac{n+1}{2} = \frac{n+1}{2}.$$

Now imagine we transform the sum A to the sum B, one step at a time, by swapping adjacent elements. Every time we do a swap of two neighboring $u < v$, the sum decreases by

$$(iu + (i+1)v) - (iv + (i+1)u) = v - u < n + 1.$$

We want to prove we land in the interval

$$I = \left[-\frac{n+1}{2}, \frac{n+1}{2} \right]$$

at some point during this transformation. But since $B \le \frac{n+1}{2} \le A$ (since $\frac{n+1}{2}$ was the average) and our step sizes were at most the length of the interval I, this is clear.