# IMO 1997/1 <br> Evan Chen 

## Twitch Solves ISL

Episode 116

## Problem

In the plane there is an infinite chessboard. For any pair of positive integers $m$ and $n$, consider a right-angled triangle with vertices at lattice points and whose legs, of lengths $m$ and $n$, lie along edges of the squares. Let $S_{1}$ be the total area of the black part of the triangle and $S_{2}$ be the total area of the white part. Let $f(m, n)=\left|S_{1}-S_{2}\right|$.
(a) Calculate $f(m, n)$ for all positive integers $m$ and $n$ which are either both even or both odd.
(b) Prove that $f(m, n) \leq \frac{1}{2} \max \{m, n\}$ for all $m$ and $n$.
(c) Show that there is no constant $C$ such that $f(m, n)<C$ for all $m$ and $n$.

## Video

https://youtu.be/zOToam5MzlM

## External Link

https://aops.com/community/p356696

## Solution

In general, we say the discrepancy of a region in the plane equals its black area minus its white area. We allow negative discrepancies, so discrepancy is additive and $f(m, n)$ equals the absolute value of the discrepancy of a right triangle with legs $m$ and $n$.

For (a), the answers are 0 and $1 / 2$ respectively. To see this, consider the figure shown below.


Notice that triangles $A P M$ and $B Q M$ are congruent, and when $m \equiv n(\bmod 2)$, their colorings actually coincide. Consequently, the discrepancy of the triangle is exactly equal to the discrepancy of $C P Q B$, which is an $m \times n / 2$ rectangle and hence equal to 0 or $1 / 2$ according to parity.

For (b), note that a triangle with legs $m$ and $n$, with $m$ even and $n$ odd, can be dissected into one right triangle with legs $m$ and $n-1$ plus a thin triangle of area $1 / 2$ which has height $m$ and base 1 . The former region has discrepancy 0 by (a), and the latter region obviously has discrepancy at most its area of $m / 2$, hence $f(m, n) \leq m / 2$ as needed. (An alternative slower approach, which requires a few cases, is to prove that two adjacent columns have at most discrepancy $1 / 2$.)

For (c), we prove:
Claim. For each $k \geq 1$, we have

$$
f(2 k, 2 k+1)=\frac{2 k-1}{6} .
$$

Proof. An illustration for $k=2$ is shown below, where we use $(0,0),(0,2 k),(2 k+1,0)$ as the three vertices.


WLOG, the upper-left square is black, as above. The $2 k$ small white triangles just below the diagonal have area sum

$$
\frac{1}{2} \cdot \frac{1}{2 k+1} \cdot \frac{1}{2 k}\left[1^{2}+2^{2}+\cdots+(2 k)^{2}\right]=\frac{4 k+1}{12}
$$

The area of the $2 k$ black polygons sums just below the diagonal to

$$
\sum_{i=1}^{2 k}\left(1-\frac{1}{2} \cdot \frac{1}{2 k+1} \cdot \frac{1}{2 k} \cdot i^{2}\right)=2 k-\frac{4 k+1}{12}=\frac{20 k-1}{12} .
$$

Finally, in the remaining $1+2+\cdots+2 k$ squares, there are $k$ more white squares than black squares. So, it follows

$$
f(2 k, 2 k+1)=\left|-k+\frac{20 k-1}{12}-\frac{4 k+1}{12}\right|=\frac{2 k-1}{6} .
$$

