IMO 1997/1

Evan Chen

Twitch Solves ISL

Episode 116

Problem

In the plane there is an infinite chessboard. For any pair of positive integers m and n, consider a right-angled triangle with vertices at lattice points and whose legs, of lengths m and n, lie along edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

(a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

Video

https://youtu.be/zOToam5MzlM

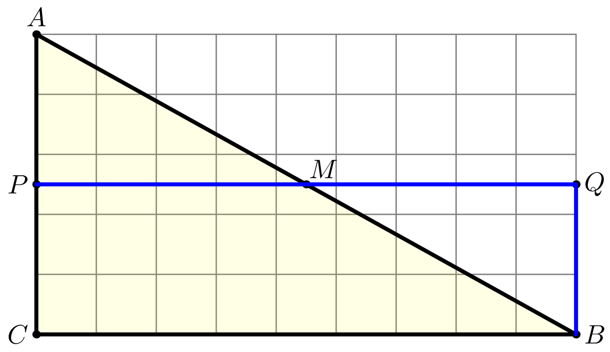
External Link

https://aops.com/community/p356696

Solution

In general, we say the *discrepancy* of a region in the plane equals its black area minus its white area. We allow negative discrepancies, so discrepancy is additive and $f(m,n)$ equals the absolute value of the discrepancy of a right triangle with legs m and n.

For (a), the answers are 0 and 1/2 respectively. To see this, consider the figure shown below.



Notice that triangles APM and BQM are congruent, and when $m \equiv n \pmod 2$, their colorings actually coincide. Consequently, the discrepancy of the triangle is exactly equal to the discrepancy of $CPQB$, which is an $m \times n/2$ rectangle and hence equal to 0 or 1/2 according to parity.

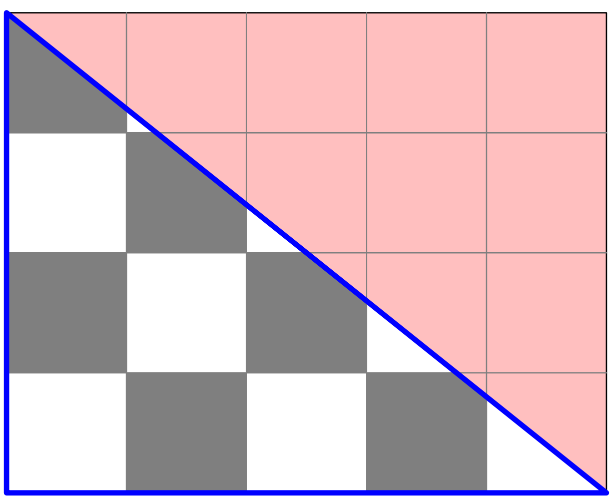
For (b), note that a triangle with legs m and n, with m even and n odd, can be dissected into one right triangle with legs m and $n-1$ plus a thin triangle of area 1/2 which has height m and base 1. The former region has discrepancy 0 by (a), and the latter region obviously has discrepancy at most its area of $m/2$, hence $f(m,n) \le m/2$ as needed. (An alternative slower approach, which requires a few cases, is to prove that two adjacent columns have at most discrepancy 1/2.)

For (c), we prove:

Claim. For each $k \ge 1$, we have

$$f(2k, 2k+1) = \frac{2k-1}{6}.$$

Proof. An illustration for $k=2$ is shown below, where we use $(0,0)$, $(0,2k)$, $(2k+1,0)$ as the three vertices.



WLOG, the upper-left square is black, as above. The $2k$ small white triangles just below the diagonal have area sum

$$\frac{1}{2}\cdot\frac{1}{2k+1}\cdot\frac{1}{2k}\left[1^2+2^2+\cdots+(2k)^2\right]=\frac{4k+1}{12}$$

The area of the $2k$ black polygons sums just below the diagonal to

$$\sum_{i=1}^{2k}\left(1-\frac{1}{2}\cdot\frac{1}{2k+1}\cdot\frac{1}{2k}\cdot i^2\right)=2k-\frac{4k+1}{12}=\frac{20k-1}{12}.$$

Finally, in the remaining $1+2+\cdots+2k$ squares, there are k more white squares than black squares. So, it follows

$$f(2k,2k+1)=\left|-k+\frac{20k-1}{12}-\frac{4k+1}{12}\right|=\frac{2k-1}{6}.$$

□