

IntBee 2023/S7

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TWITCH SOLVES ISL

Episode 114

Problem

Evaluate the indefinite integral

$$\int (\sqrt{x+1} - \sqrt{x})^\pi dx.$$

Video

<https://youtu.be/Sk106c0BbV8>

Solution

Note that

$$\frac{d}{dx} (\sqrt{x+1} - \sqrt{x}) = \frac{1}{2} \left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}} \right) = -\frac{\sqrt{x+1} - \sqrt{x}}{2\sqrt{x(x+1)}}.$$

Define

$$\begin{aligned} u &:= \sqrt{x+1} - \sqrt{x} \implies \frac{1}{u} = \sqrt{x+1} + \sqrt{x} \\ \implies u^{-2} - u^2 &= 4\sqrt{x(x+1)}. \end{aligned}$$

Differentiating u with respect to x ,

$$\begin{aligned} \frac{du}{dx} &= -\frac{1}{2}u \cdot \frac{1}{\sqrt{x(x+1)}} \\ &= -\frac{1}{2}u \cdot \frac{4}{u^{-2} - u^2} \\ dx &= \frac{u^2 - u^{-2}}{2u} du. \end{aligned}$$

Thus the original integral can be resolved with a u -substitution:

$$\begin{aligned} \int \left(\frac{u^2 - u^{-2}}{2u} \right) \cdot u^\pi du &= \frac{1}{2} \int (u^{\pi+1} - u^{\pi-3}) du \\ &= \frac{1}{2} \left(\frac{u^{\pi+2}}{\pi+2} - \frac{u^{\pi-2}}{\pi-2} \right) + C \\ &= \boxed{\frac{1}{2} \left(\frac{(\sqrt{x+1} - \sqrt{x})^{\pi+2}}{\pi+2} - \frac{(\sqrt{x+1} - \sqrt{x})^{\pi-2}}{\pi-2} \right) + C}. \end{aligned}$$