

# **Twitch 113.2**

**Evan Chen**

TWITCH SOLVES ISL

Episode 113

## **Problem**

Let  $f: [0, 1] \rightarrow [0, 1]$  be monotonically increasing. Show that there exists  $\xi \in [0, 1]$  with  $f(\xi) = \xi$ .

## **Video**

<https://youtu.be/mUDShxX8XNk>

## Solution

Assume not. Then

$$\begin{aligned} A &:= \{f(x) < x\} \ni 1 \\ B &:= \{f(x) > x\} \ni 0 \end{aligned}$$

This gives a partition  $A \sqcup B = [0, 1]$ .

Let  $\xi = \sup B$ . We consider two cases:

- If  $\xi \in B$ , then  $f(a) < a$  for all  $a > \xi$ . Hence

$$f(\xi) < f(a) < a \quad \forall a > \xi$$

so  $f(\xi) \leq \xi$ , contradiction to  $\xi \in B$ .

- Otherwise, for all  $\varepsilon > 0$ , the set  $(\xi - \varepsilon, \xi) \cap B$  is nonempty. So

$$f(\xi) > f\left(\text{something in } (\xi - \varepsilon, \xi) \cap B\right) > \xi - \varepsilon$$

for all  $\varepsilon > 0$ , ergo  $f(\xi) \leq \xi$ . This violates the assumption  $\xi \in A$ .

Since both cases give a contradiction, we're done.