Twitch 113.2

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Twitch Solves ISL

Episode 113

Problem

Let $f\colon [0,1] \to [0,1]$ be monotonically increasing. Show that there exists $\xi \in [0,1]$ with $f(\xi) = \xi$.

Video

https://youtu.be/mUDShxX8XNk

Solution

Assume not. Then

$$A \coloneqq \{f(x) < x\} \ni 1$$
$$B \coloneqq \{f(x) > x\} \ni 0$$

This gives a partition $A \sqcup B = [0,1]$.

Let $\xi = \sup B$. We consider two cases:

- If $\xi \in B$, then $f(a) < a$ for all $a > \xi$. Hence

  $$f(\xi) < f(a) < a \qquad \forall a > \xi$$

  so $f(\xi) \le \xi$, contradiction to $\xi \in B$.

- Otherwise, for all $\varepsilon > 0$, the set $(\xi - \varepsilon, \xi) \cap B$ is nonempty. So

  $$f(\xi) > f\Big(\text{something in } (\xi - \varepsilon, \xi) \cap B\Big) > \xi - \varepsilon$$

  for all $\varepsilon > 0$, ergo $f(\xi) \le \xi$. This violates the assumption $\xi \in A$.

Since both cases give a contradiction, we're done.