GaussMO 2022/3

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Twitch Solves ISL

Episode 113

Problem

On a triangle $\triangle ABC$ let ω be its circumcircle and let H_A be a point on ω such that $AH_A \perp BC$. Let the projections from B, C to AC, AB be E, F respectively. Let H_AE, H_AF meet ω again at P, Q respectively and let PQ hit BC at T. Show that one of the tangents from T to ω bisects the common chord of the circumcircles of triangles BFP and CEQ.

Video

https://youtu.be/RsTDs5YNlIM

External Link

https://aops.com/community/p26134873

Solution

Let K be a point such that $ABKC$ is a cyclic harmonic quadrilateral. Let M be the midpoint of $\overline{EF}$.

Claim (ELMO SL 2019 G1)**.** Lines AK, BP, and CQ meet at M.

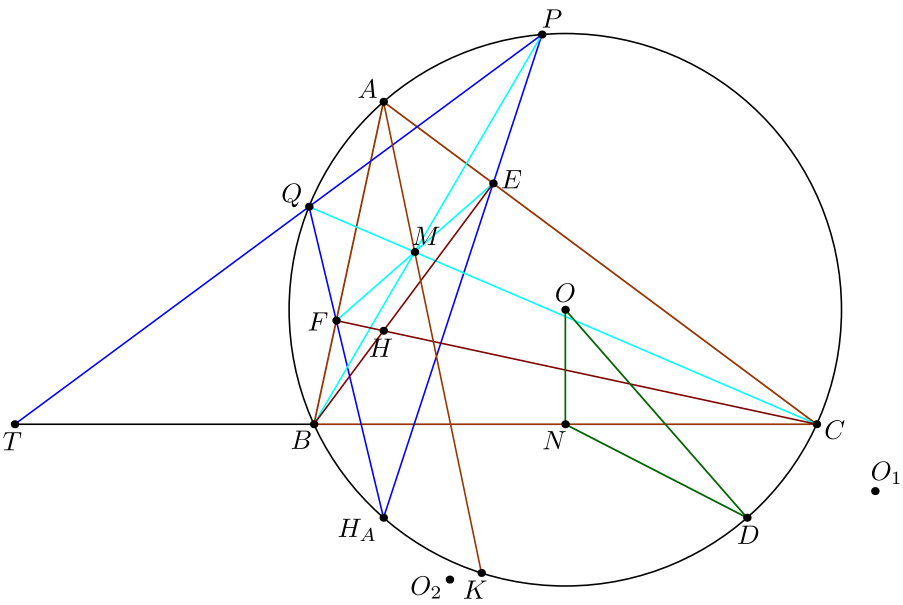
Proof. The fact that $\overline{AKM}$ collinear is well-known, as is the lemma that $(BFMK)$ and $(CEMK)$ are cyclic.

We also contend that EFH_AK is cyclic. Indeed, $\overline{EF}$ and $\overline{H_AK}$ concur at the harmonic conjugate of $\overline{AH} \cap \overline{BC}$ with respect to $\overline{BC}$, so this follows by radical axis.

Then, we do the angle chase

$$\measuredangle KBM = \measuredangle KFM = \measuredangle KFE = \measuredangle KH_AE = \measuredangle KH_AP = \measuredangle KBP$$

to get that B, M, P are collinear. □



We tie in the radical axis of the two given circles now.

Claim. Line $\overline{AMK}$ is both the radical axis of the given circles and exactly the polar of T.

Proof. Continuing on, we have

$$-1 = (AK;BC)_\omega \stackrel{M}{=} (KA;PQ)$$

and that's enough to imply that $T = \overline{PQ} \cap \overline{BC} \cap \overline{AA} \cap \overline{KK}$ with respect to ω. As A and M have equal power to (BFP) and (CEQ), □

We will now prove K is the midpoint of the common chord; this will solve the problem (since $\overline{TK}$ is tangent). Define N as the midpoint of $\overline{BC}$ and D as the antipode.

Claim. Triangles $\triangle AMC$ and $\triangle OND$ are orthologic.

Proof. The altitude from A to $\overline{ND}$ is well-known to pass through $\overline{EF} \cap \overline{BC}$. The altitude from C to $\overline{ON}$ is $\overline{BC}$ itself, and the altitude from M to $\overline{OD}$ is exactly line $\overline{EF}$ (since $\overline{AOD} \perp \overline{EF}$). So $\overline{EF} \cap \overline{BC}$ is the orthology center. □

But running the orthology the other way, the perpendicular from O to $\overline{MC}$ and the perpendicular from N to $\overline{AC}$ are exactly the perpendicular bisectors of $\overline{QC}$ and $\overline{CE}$, and meet at the circumcenter O_2 of $\triangle CEQ$, say. Meanwhile, $\overline{DK} \perp \overline{AM}$. In other words, $\overline{DO_2K} \perp \overline{AMK}$, and we're done.