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Twitch Solves ISL

Episode 113

Problem

Find all pairs of positive integers (n, d) such that the following statement holds: if x is a positive integer for which 2^{x+n} and 5^x have the same first decimal digit, that digit is equal to d. (In particular, $d \leq 9$.)

Video

https://youtu.be/O2-1qI9SEHg

Solution

Answer: $(2,6)$, $(4,1)$, and $(6,2)$.

First, note that

$$2^{x+n} \cdot 5^x = 2^n \cdot 10^x.$$

To show these work, we start with the wonderful table that shows what happens when one multiplies two numbers with the same starting digit:

- $\overline{01\cdots} \le \overline{1\cdots} \cdot \overline{1\cdots} < \overline{04\cdots}$
- $\overline{04\cdots} \le \overline{2\cdots} \cdot \overline{2\cdots} < \overline{09\cdots}$
- $\overline{09\cdots} \le \overline{3\cdots} \cdot \overline{3\cdots} < \overline{16\cdots}$
- $\overline{16\cdots} \le \overline{4\cdots} \cdot \overline{4\cdots} < \overline{25\cdots}$
- $\overline{25\cdots} \le \overline{5\cdots} \cdot \overline{5\cdots} < \overline{36\cdots}$
- $\overline{36\cdots} \le \overline{6\cdots} \cdot \overline{6\cdots} < \overline{49\cdots}$
- $\overline{49\cdots} \le \overline{7\cdots} \cdot \overline{7\cdots} < \overline{64\cdots}$
- $\overline{64\cdots} \le \overline{8\cdots} \cdot \overline{8\cdots} < \overline{81\cdots}$
- $\overline{81\cdots} \le \overline{9\cdots} \cdot \overline{9\cdots} < \overline{100\cdots}$

Hence, if two numbers with the same leading digit have product $4 \cdot 10^x$, for example, that leading digit is 2 or 6. Moreover, the former case only occurs if the two numbers are both exactly of the form $200\cdots00$. So in the situation of the problem, if $2^{x+2} \cdot 5^x = 4 \cdot 10^x$ and 2^{x+2} and 5^x have the same leading digit, that leading digit must be exactly 6. This shows that $(2,6)$ is a working pair. The pairs $(4,1)$ and $(6,2)$ have exactly the same proof.

The converse is painful. It relies on the following lemma:

Lemma (Not JMO 2016/2)**.** For any finite string of decimal digits, some power of 5 begins with that string of digits.

Proof. The idea is to note that $\log_{10} 5$ is irrational, and hence for any nontrivial interval $I \subseteq [0,1]$ there exists an integer k such that the fractional part of $k \log_{10} 5 = \log_{10}(5^k)$ lies in I. (This is sometimes called Kronecker approximation theorem.) $\square$

Now do some work: if 2^n is not a one- or two-digit perfect square, there are two entries in the table earlier for which the first two digits line up with the first two digits of $2^n = \overline{d_1 d_2 \dots}$. Then use Kronecker approximation to rig 5^x such that it starts with some integer s for which $s \approx \sqrt{d_1 d_2 . d_3 d_4 d_5 \dots}$, and then again with $s \approx \sqrt{d_1 . d_2 d_3 d_4 \dots}$. (This is admittedly sketchy.)