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Twitch Solves ISL

Episode 113

Problem

Find all pairs of positive integers (n, d) such that the following statement holds: if x is a positive integer for which 2^{x+n} and 5^x have the same first decimal digit, that digit is equal to d. (In particular, $d \leq 9$.)

Video

https://youtu.be/O2-1qI9SEHg

Solution

Answer: $(2,6)$, $(4,1)$, and $(6,2)$.

First, note that

$$2^{x+n} \cdot 5^x = 2^n \cdot 10^x.$$

To show these work, we start with the wonderful table that shows what happens when one multiplies two numbers with the same starting digit:

- $\overline{01\ldots} \le \overline{1\ldots} \cdot \overline{1\ldots} < \overline{04\ldots}$
- $\overline{04\ldots} \le \overline{2\ldots} \cdot \overline{2\ldots} < \overline{09\ldots}$
- $\overline{09\ldots} \le \overline{3\ldots} \cdot \overline{3\ldots} < \overline{16\ldots}$
- $\overline{16\ldots} \le \overline{4\ldots} \cdot \overline{4\ldots} < \overline{25\ldots}$
- $\overline{25\ldots} \le \overline{5\ldots} \cdot \overline{5\ldots} < \overline{36\ldots}$
- $\overline{36\ldots} \le \overline{6\ldots} \cdot \overline{6\ldots} < \overline{49\ldots}$
- $\overline{49\ldots} \le \overline{7\ldots} \cdot \overline{7\ldots} < \overline{64\ldots}$
- $\overline{64\ldots} \le \overline{8\ldots} \cdot \overline{8\ldots} < \overline{81\ldots}$
- $\overline{81\ldots} \le \overline{9\ldots} \cdot \overline{9\ldots} < \overline{100\ldots}$

Hence, if two numbers with the same leading digit have product $4 \cdot 10^x$, for example, that leading digit is 2 or 6. Moreover, the former case only occurs if the two numbers are both exactly of the form $200\ldots00$. So in the situation of the problem, if $2^{x+2} \cdot 5^x = 4 \cdot 10^x$ and 2^{x+2} and 5^x have the same leading digit, that leading digit must be exactly 6. This shows that $(2,6)$ is a working pair. The pairs $(4,1)$ and $(6,2)$ have exactly the same proof.

The converse is painful. It relies on the following lemma:

Lemma (Not JMO 2016/2)**.** For any finite string of decimal digits, some power of 5 begins with that string of digits.

Proof. The idea is to note that $\log_{10} 5$ is irrational, and hence for any nontrivial interval $I \subseteq [0,1]$ there exists an integer k such that the fractional part of $k \log_{10} 5 = \log_{10}(5^k)$ lies in I. (This is sometimes called Kronecker approximation theorem.) $\square$

Now do some work: if 2^n is not a one- or two-digit perfect square, there are two entries in the table earlier for which the first two digits line up with the first two digits of $2^n = \overline{d_1 d_2 \ldots}$. Then use Kronecker approximation to rig 5^x such that it starts with some integer s for which $s \approx \sqrt{d_1 d_2.d_3 d_4 d_5 \ldots}$, and then again with $s \approx \sqrt{d_1.d_2 d_3 d_4 \ldots}$. (This is admittedly sketchy.)