

INMO 2023/6

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TWITCH SOLVES ISL

Episode 112

Problem

Euclid has a tool called *cyclos* which allows him to do the following:

- Given three non-collinear marked points, draw the circle passing through them.
- Given two marked points, draw the circle with them as endpoints of a diameter.
- Mark any intersection points of two drawn circles or mark a new point on a drawn circle.

Show that given two marked points, Euclid can draw a circle centered at one of them and passing through the other, using only the *cyclos*.

Video

<https://youtu.be/kR3Tzw5JIUU>

External Link

<https://aops.com/community/p26888633>

Solution

We start with the following lemmas.

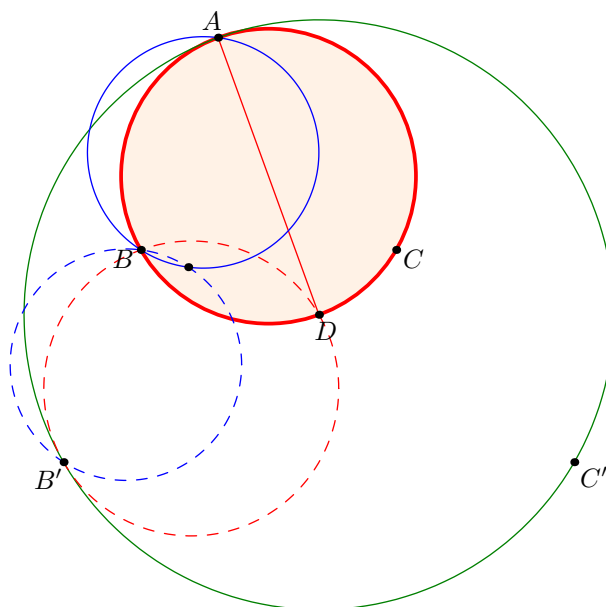
Observe first that given any non-right nondegenerate triangle ABC :

- Draw (AB) , (BC) , (CA) to get feet D , E , F .
- Draw (AEF) , (BFD) , (CDE) to get the orthocenter H .
- Note also that (BCH) is the reflection of (ABC) across \overline{BC} . So, we may reflect a circle over a chord.

Suppose we're given to start a circle γ with diameter \overline{AD} . Pick points B and C on γ so that ABC is acute.

Claim. We may construct get the reflection B' of A across B .

Proof. Let δ_B be any circle through A and B other than γ or (AB) (for example one may let the third vertex be the foot from the orthocenter of ABC to line BD). Reflect both δ_B and γ about \overline{BD} . \square



Define C' similarly. Then $(AB'C')$ is the desired circle by homothety.