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TWITCH SOLVES ISL

Episode 112

Problem

Euclid has a tool called *cyclos* which allows him to do the following:

- Given three non-collinear marked points, draw the circle passing through them.

- Given two marked points, draw the circle with them as endpoints of a diameter.

- Mark any intersection points of two drawn circles or mark a new point on a drawn circle.

Show that given two marked points, Euclid can draw a circle centered at one of them and passing through the other, using only the cyclos.

Video

External Link

Solution

We start with the following lemmas.

Observe first that given any non-right nondegenerate triangle ABC:

- Draw (AB), (BC), (CA) to get feet D, E, F.

- Draw (AEF), (BFD), (CDE) to get the orthocenter H.

- Note also that (BCH) is the reflection of (ABC) across $\overline{BC}$. So, we may reflect a circle over a chord.

Suppose we're given to start a circle γ with diameter $\overline{AD}$. Pick points B and C on γ so that ABC is acute.

Claim. We may construct get the reflection B' of A across B.

Proof. Let δ_B be any circle through A and B other than γ or (AB) (for example one may let the third vertex be the foot from the orthocenter of ABC to line BD). Reflect both δ_B and γ about $\overline{BD}$. $\qquad\square$
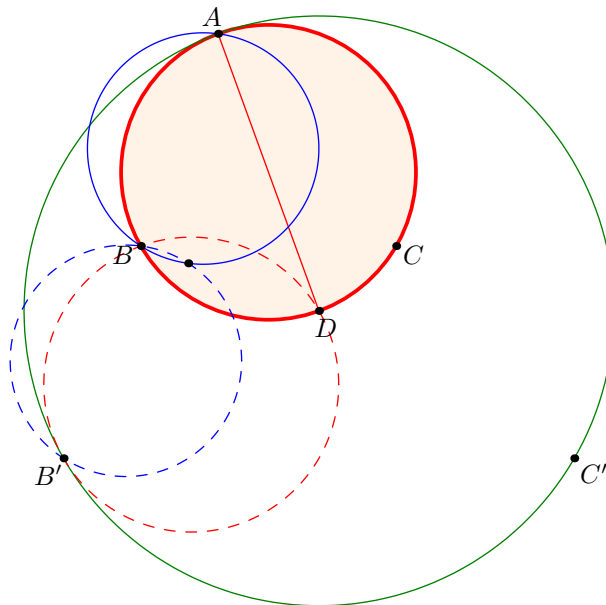
Define C' similarly. Then $(AB'C')$ is the desired circle by homothety.